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On a Singular Degenerate Reaction-Diffusion Model Applied to Quenching and Biology

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Abstract

In this paper, we are interested in studying a singular nonlinear degenerate reaction-diffusion model having a natural growth with respect to the gradient. Our approach uses Schauder's fixed point theorem. This type of problem has numerous important applications across multiple disciplines, such as biology, ecology and medicine. By employing rigorous mathematical techniques, we aim to advance the theoretical understanding of this type of nonlinear degenerate reaction-diffusion problems and lay the groundwork for further developments and real-world implementations.

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Keywords: Reaction-diffusion equations, degenerate parabolic equations, singular parabolic equations, natural growth term.

1 Introduction

Singular degenerated reaction-diffusion systems are a class of partial differential equations that exhibit a unique mathematical structure. These systems are characterized by the presence of a singular or degenerate term in the equation, which can lead to complex behavior and interesting applications. In the context of quenching, singular degenerated reaction-diffusion

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systems can be used to model phenomena such as the quenching of flames or the extinction of biological populations. The singular term in the equation can represent a critical threshold or a phase transition, where the system's behavior changes dramatically. For example, consider a model for the spread of a fire in a forest. The reaction-diffusion equation can describe the dynamics of the fire, with the singular term representing the critical temperature at which the fire is extinguished. By analyzing the properties of the singular degenerated reaction-diffusion system, researchers can gain insights into the conditions under which the fire will be quenched and the factors that influence the quenching process. Similarly, in the context of biology, singular degenerated reaction-diffusion systems can be used to model the dynamics of biological populations, such as the spread of infectious diseases or the growth of tumors. The singular term in the equation can represent a critical population density or a phase transition in the system's behavior, such as the onset of extinction or the transition to a state of uncontrolled growth. The analysis of singular degenerated reaction-diffusion systems often involves techniques from nonlinear analysis, such as the study of free boundaries, the analysis of steady-state solutions, and the investigation of the asymptotic behavior of the solutions. These systems can exhibit rich and complex dynamics, including the formation of patterns, the occurrence of bifurcations, and the existence of multiple stable states. The applications of singular degenerated reaction-diffusion systems span various fields, including ecology, epidemiology, materials science, and even social sciences. By understanding the mathematical properties of these systems and their connection to real-world phenomena, researchers can develop more accurate models and gain insights that inform practical applications. We find numerous real applications in biology, medicine and ecology in the works of Mesbahi et *al.* Mesbahi et *al.* [10, 11, 18, 19, 20], and also in DiBendetto [9], Murray [21, 22] and corresponding references therein.

The problem we will study here fits into this context, and we find that it has many important applications in biology, medicine, the environment, and many other interesting scientific fields. The analysis of these systems often involves advanced mathematical techniques. By understanding the rich and complex dynamics exhibited by these systems, researchers can develop more accurate models that provide valuable insights in various scientific domains. We are interested in the following nonlinear singular degenerated reaction-diffusion system having natural growth with respect to the gradient

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathbf{div}\ \left(a\left(t,x\right)\nabla u\right) + d\left(t,x\right)\dfrac{\left|\nabla u\right|^p}{u^\gamma} = f\left(t,x\right) & \text{in } Q_T \\ u\left(t,x\right) = 0 & \text{on } \Gamma \\ u\left(0,x\right) = u_0\left(x\right) & \text{in } \Omega \end{cases} \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N > 2$, and Q is the cylinder $(0,T) \times \Omega$, $T > 0$, $\Gamma = (0,T) \times \partial\Omega$, $2 < p < N$, $0 < \gamma < 1$, $a\left(t,x\right)$ and $d\left(t,x\right)$ are two bounded measurable functions satisfying

$$0 < \alpha_1 \leq a\left(t,x\right) \leq \alpha_2 \tag{2}$$

$$0 < \beta_1 \leq d\left(t,x\right) \leq \beta_2 \tag{3}$$

where α_1, α_2, β_1 and β_2 are fixed real numbers such that $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$. On the function f, we assume that it is non-negative and not identically zero, and that it belongs to the Lebesgue space $L^m\left(Q_T\right)$ with $m > 1$. Moreover, the initial data $u_0 \in L^\infty\left(\Omega\right)$ satisfies the following condition of strict positivity

$$\exists D_\omega > 0, \forall\omega \subset\subset \Omega : u_0 \geq D_\omega$$

We find a detailed history of this problem and its applications in numerous areas in Benkirane et *al.* [2], Boccardo et *al.* [3], Dall'Aglio et *al.* [5, 6, 7], De Bonis and Giachetti [8], El Hadfi et *al.* [12], El Ouardy and El Hadfi [13], Keller and Choen [15], Magliocca [16], Martínez-Aparicio and Petitta [17], Nachman and Callegari [23], Youssfi et *al.* [25] and references therein.

The problem described in (1) presents several key difficulties that must be addressed. Firstly, the presence of the lower-order term introduces complications. The natural growth term in the equation depends on the gradient, which adds mathematical complexity. Additionally, the singularity in the equation depends on the variable u, further complicating the analysis. Perhaps most challenging is the need to prove the strict positivity of the solution within the interior of the parabolic cylinder. Establishing this positivity property is a non-trivial task. To overcome these difficulties, we must approximate the singular problem (1) by another non-singular one, and we show that this problem admits a non-negative solution (the proof is based on the application of Schauder's fixed point theorem) and that this solution is strictly positive in the interior of the parabolic cylinder (the proof is based on the use of the intrinsic Harnack inequality).

Below, we will use $|E|$ to denote the Lebesgue measure of a subset $E \subset \mathbb{R}^N$. The Hölder conjugate exponent of $q > 1$ is $q' = \frac{q}{q-1}$, while the Sobolev conjugate exponent of p for $1 \leq p < N$ is $\frac{Np}{N-p}$. For a fixed $k > 0$, we define the truncation function T_k as follows:

$$T_k(s) = \max\{-k, \min\{s, k\}\}$$

To streamline notation, we will use C to represent values that may vary from step to step or even within the same step, depending on some parameters. However, C will remain constant with respect to the indices of any sequences introduced.

2 Statement of the main result

First, we have to clarify in which sense we want to solve our problem.

Definition 1 *A weak solution to problem* (1) *is a function* $u \in L^1\left(0, T; W_0^{1,1}(\Omega)\right)$ *such that for every* $\omega \subset\subset \Omega$ *there exists* c_ω *such that* $u \geq c_\omega > 0$ *in* $\omega \times (0, T)$, $\left(a(t,x)|\nabla u|^{p-1}\right) \in L^1(Q_T)$, $\dfrac{|\nabla u|^p}{u^\gamma} \in L^1\left(0, T; L^1_{loc}(\Omega)\right)$. *Furthermore, we have that*

$$-\int_{Q_T} u\frac{\partial \phi}{\partial t} dtdx + \int_{Q_T} a(t,x)\nabla u.\nabla \phi dtdx$$
$$+\int_{Q_T} d(t,x)\frac{|\nabla u|^p}{u^\gamma}\phi dtdx = \int_{Q_T} f\phi dtdx + \int_\Omega u_0(x)\phi(0,x)$$

for every $\phi \in C_c^1([0,T) \times \Omega)$.

Now, we can state the main result of this paper, it is the following theorem.

Theorem 2 *Let* $0 < \gamma < 1$, $\lambda = \dfrac{p(N+1+\delta)}{p(N+1+\delta) - N\delta}$ *and* $\delta = \min\{\gamma, 1-q\}$. *Assume that a satisfy* (2), *d satisfy* (3) *and* $f \in L^m(Q_T)$ *with* $1 < m < \frac{N}{p+1}$. *Then there exists a solution u of problem* (1) *in the sense of Definition 1 verify the following regularity:*

(i) If $\lambda \leq m < \dfrac{N}{p+1}$, *then* $u \in L^p\left(0, T; W_0^{1,p}(\Omega)\right) \cap L^\sigma(Q_T)$ *with*

$$\sigma = m\frac{N(p-\delta)+p}{N-pm+p}$$

(ii) If $1 < m < \lambda$, *then* $u \in L^s\left(0, T; W_0^{1,s}(\Omega)\right) \cap L^\sigma(Q_T)$ *with*

$$s = m\frac{N(p-\delta)+p}{N+1-\delta(m-1)}$$

3 Approximating Scheme

Let $0 < \varepsilon < 1$. We approximate problem (1) by the following nonlinear and non-singular problem

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \mathbf{div}\ (a(t,x)\nabla u_\varepsilon) + d(t,x)\dfrac{u_\varepsilon |\nabla u_\varepsilon|^p}{(\varepsilon + |u_\varepsilon|)^{\gamma+1}} = f_\varepsilon(t,x) & \text{in } Q_T \\ u_\varepsilon(t,x) = 0 & \text{on } \Gamma \\ u_\varepsilon(0,x) = u_{\varepsilon 0}(x) & \text{in } \Omega \end{cases} \tag{4}$$

where

$$f_\varepsilon = \frac{f}{1+\varepsilon f} \quad \text{and } f_\varepsilon \in L^\infty(Q_T)$$

such that

$$\|f_\varepsilon\|_{L^m(Q_T)} \le \|f\|_{L^m(Q_T)} \quad \text{and} \quad f_\varepsilon \to f \text{ strongly in } L^m(Q_T),\ m > 1 \tag{5}$$

and

$$u_{\varepsilon 0}(x) = \frac{u_0(x)}{1+\varepsilon u_0(x)} \in L^\infty(\Omega)$$

such that

$$\|u_{\varepsilon 0}\|_{L^\infty(\Omega)} \le \|u_\varepsilon\|_{L^\infty(\Omega)} \quad \text{and } u_{\varepsilon 0} \to u_0 \text{ strongly in } L^1(\Omega)$$

Problem (4) admits weak solutions $u_\varepsilon \in L^p\left(0,T;W_0^{1,p}(\Omega)\right) \cap L^\infty(Q_T)$, as shown in references Dall'Aglio and Orsina [4] and Lions [14]. Additionally, the solution of problem (4) is continuous in time, meaning $u_\varepsilon \in C\left([0,T];L^1_{loc}(\Omega)\right)$. Since the right side of (4) is non-negative, u_ε is also non-negative.

Lemma 3 *Let u_ε be solutions to problem* (4)*. Then*

$$\int_{Q_T} d(t,x)\frac{u_\varepsilon |\nabla u_\varepsilon|^p}{(\varepsilon + |u_\varepsilon|)^{\gamma+1}} \le |Q_T|^{1-\frac{1}{m}} \|f\|_{L^m(Q_T)} + \|u_0\|_{L^1(\Omega)}$$

Proof. See DiBendetto [9]. ■

Remark 4 *According to Lemma 3, and since*

$$\int_{Q_T} d(t,x)\frac{u_\varepsilon |\nabla u_\varepsilon|^p}{(\varepsilon + |u_\varepsilon|)^{\gamma+1}} \ge 0 \quad \textit{for} \quad f \in L^1(Q_T)$$

one has that

$$\int_{Q_T} \left| d(t,x)\frac{u_\varepsilon |\nabla u_\varepsilon|^p}{(\varepsilon + |u_\varepsilon|)^{\gamma+1}} - f \right| \le 2|Q_T|^{1-\frac{1}{m}} \|f\|_{L^m(Q_T)} + \|u_0\|_{L^1(\Omega)} < C$$

Lemma 5 *Let the assumptions of Theorem 2 be in force. Then the solution u_ε of (4) satisfy the following estimate:*

(i) If $\lambda \leq m < \dfrac{N}{p+1}$, then u_ε is uniformly bounded in the space

$$L^p\left(0,T;W_0^{1,p}(\Omega)\right) \cap L^\sigma(Q_T)$$

(ii) If $1 < m < \lambda$, then u_ε is uniformly bounded in the space

$$L^s\left(0,T;W_0^{1,s}(\Omega)\right) \cap L^\sigma(Q_T)$$

where s and σ are defined in Theorem 2.

4 Proof of the main result

Now we can prove Theorem 2.
Proof of Theorem 2. In view of Lemma 5,

$$u_\varepsilon \rightharpoonup u \text{ weakly in } L^\delta\left(0,T;W_0^{1,\delta}(\Omega)\right), \ \forall \delta < s < p \text{ and a.e. in } Q_T$$

By Remark 4, $f_\varepsilon - d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \in L^1(Q_T)$ and from Lemma 5, we have $a(t,x)\nabla u_\varepsilon$ is bounded in $L^\rho(Q_T)$ for all $1 \leq \rho < \frac{s}{p-1} < p$. Then **div** $(a(t,x)\nabla u_\varepsilon)$ is bounded in the space $L^{\rho'}(Q_T) \subset L^{p'}(Q_T) \subset L^{p'}\left(0,T;W_0^{-1,p'}(\Omega)\right)$, and then $\frac{\partial u_\varepsilon}{\partial t}$ is bounded in the space $L^{p'}\left(0,T;W_0^{-1,p'}(\Omega)\right)+L^1(Q_T)$. Using the compactness results in Simon [24], we obtain

$$u_\varepsilon \to u \text{ strongly in } L^1(Q_T) \text{ and a.e. in } Q_T. \tag{6}$$

We can use the same proof as in Abdellaoui and Redwane [1], we obtain

$$T_k(u_\varepsilon) \to T_k(u) \text{ strongly in } L^p\left(0,T;W_0^{1,p}(\Omega)\right)$$

and also we have

$$\nabla u_\varepsilon \to \nabla u \text{ in a.e. in } Q_T \tag{7}$$

On the other hand, recalling (2), (6), (7), Lemma 5 and the dominated convergence theorem implies that the sequence $a(t,x)\nabla u_\varepsilon$ converges weakly to $a(t,x)\nabla u$ in $L^\rho(Q_T)$ for every $1 \leq \rho < \frac{s}{p-1}$. Therefore, for every $\varphi \in C_c^1(\Omega \times [0,T))$,

$$\lim_{\varepsilon\to 0}\int_Q a(t,x)\nabla u_\varepsilon \nabla\varphi = \int_Q a(t,x)\nabla u \nabla\varphi \tag{8}$$

Now we prove that

$$d(t,x)\,\frac{u_\varepsilon\,|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \to d(t,x)\,\frac{|\nabla u|^p}{u^\gamma} \quad \text{, strongly locally in } L^1(Q_T)$$

For any measurable compact subset E of Q_T, we have

$$\begin{aligned}\int_E d(t,x)\,\frac{u_\varepsilon\,|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} &= \int_{E\cap\{u_\varepsilon\le k\}} d(t,x)\,\frac{u_\varepsilon\,|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \\ &\quad + \int_{E\cap\{u_\varepsilon> k\}} d(t,x)\,\frac{u_\varepsilon\,|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \\ &\le \int_{E\cap\{u_\varepsilon\le k\}} d(t,x)\,\frac{|\nabla u_\varepsilon|^p}{u_\varepsilon^\gamma} \\ &\quad + \int_{E\cap\{u_\varepsilon> k\}} d(t,x)\,\frac{u_\varepsilon\,|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}\end{aligned}$$

We get

$$\int_E d(t,x)\,\frac{u_\varepsilon\,|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \le \frac{1}{c_\varepsilon^\gamma}\int_E d(t,x)\,|\nabla T_k(u_\varepsilon)|^p + \int_{E\cap\{u_\varepsilon> k\}} d(t,x)\,\frac{u_\varepsilon\,|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}$$

Let $\nu>0$ be fixed. For $k>1$, we use $T_1(u_\varepsilon - T_{k-1}(u_\varepsilon))$ as a test function in (4), yielding

$$\begin{aligned}&\int_0^T\int_\Omega \frac{\partial u_\varepsilon}{\partial t} T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) + \int_{Q_T} a(t,x)\,\nabla u_n T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) \\ &+ \int_{Q_T} d(t,x)\,\frac{u_\varepsilon\,|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) \\ =\ & \int_{Q_T} f_\varepsilon T_1(u_\varepsilon - T_{k-1}(u_\varepsilon))\end{aligned}$$

Recalling (2) and the fact $u_\varepsilon\ge 0$, we can write

$$\begin{aligned}&\int_\Omega S_1(u_\varepsilon(T)) + \alpha_1\int_{\{k-1\le u_\varepsilon\le k\}} |\nabla u_\varepsilon|^p \\ &+ \int_{Q_T} d(t,x)\,\frac{u_\varepsilon\,|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) \\ \le\ & \int_{Q_T} f_\varepsilon T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) + \int_{Q_T} S_1(u_0),\end{aligned}$$

where

$$S_1\left(u_\varepsilon(T)\right)=\int_0^{u_\varepsilon(T)} T_1\left(s-T_{k-1}(s)\right)ds.$$

It is easy to see that $S_1\left(u_\varepsilon(T)\right)\geq 0$ a.e. in Ω. After the first and second non-negative terms of the previous inequality are removed, we arrive at

$$\begin{aligned}
&\int_{Q_T} d(t,x)\frac{u_\varepsilon\left|\nabla u_\varepsilon\right|^p}{\left(u_\varepsilon+\varepsilon\right)^{\gamma+1}}T_1\left(u_\varepsilon-T_{k-1}(u_\varepsilon)\right) \qquad (9)\\
\leq\ & \int_{Q_T} f_\varepsilon T_1\left(u_\varepsilon-T_{k-1}(u_\varepsilon)\right)+\int_{Q_T} S_1(u_0)\\
=\ & \int_{Q_T} f_\varepsilon T_1\left(u_\varepsilon-T_{k-1}(u_\varepsilon)\right)\int_\Omega\int_0^{u_0} T_1\left(s-T_{k-1}(s)\right)ds.
\end{aligned}$$

Since $T_1\left(u_\varepsilon-T_{k-1}(u_\varepsilon)\right)\geq 0$,

$$T_1\left(u_\varepsilon-T_{k-1}(u_\varepsilon)\right)=\begin{cases}0 & \text{if } u_\varepsilon\leq k-1\\ 1 & \text{if } u_\varepsilon>k\end{cases}$$

recalling the condition (3) and the fact that $u_\varepsilon>0$, we have

$$\begin{aligned}
&\int_{Q_T} d(t,x)\frac{u_\varepsilon\left|\nabla u_\varepsilon\right|^p}{\left(u_\varepsilon+\varepsilon\right)^{\gamma+1}}T_1\left(u_\varepsilon-T_{k-1}(u_\varepsilon)\right)\\
=\ & \int_{Q_T\cap\{u_\varepsilon>k\}} d(t,x)\frac{u_\varepsilon\left|\nabla u_\varepsilon\right|^p}{\left(u_\varepsilon+\varepsilon\right)^{\gamma+1}}T_1\left(u_\varepsilon-T_{k-1}(u_\varepsilon)\right)\\
&+\int_{Q_T\cap\{u_\varepsilon\leq k\}} d(t,x)\frac{u_\varepsilon\left|\nabla u_\varepsilon\right|^p}{\left(u_\varepsilon+\varepsilon\right)^{\gamma+1}}T_1\left(u_\varepsilon-T_{k-1}(u_\varepsilon)\right)\\
=\ & \int_{Q_T\cap\{u_\varepsilon>k\}} d(t,x)\frac{u_\varepsilon\left|\nabla u_\varepsilon\right|^p}{\left(u_\varepsilon+\varepsilon\right)^{\gamma+1}}\\
&+\int_{Q\cap\{u_\varepsilon\leq k\}} d(t,x)\frac{u_\varepsilon\left|\nabla u_\varepsilon\right|^p}{\left(u_\varepsilon+\varepsilon\right)^{\gamma+1}}T_1\left(u_\varepsilon-T_{k-1}(u_\varepsilon)\right)\\
\geq\ & \int_{E\cap\{u_\varepsilon>k\}} d(t,x)\frac{u_\varepsilon\left|\nabla u_\varepsilon\right|^p}{\left(u_\varepsilon+\varepsilon\right)^{\gamma+1}}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{Q_T} f_\varepsilon T_1\left(u_\varepsilon - T_{k-1}\left(u_\varepsilon\right)\right) \\
= &\int_{Q_T\cap\{u_\varepsilon\leq k-1\}} f_\varepsilon T_1\left(u_\varepsilon - T_{k-1}\left(u_\varepsilon\right)\right) + \int_{Q_T\cap\{k-1<u_\varepsilon\leq k\}} f_\varepsilon T_1\left(u_\varepsilon - T_{k-1}\left(u_\varepsilon\right)\right) \\
&+ \int_{Q\cap\{u_\varepsilon>k\}} f_\varepsilon T_1\left(u_\varepsilon - T_{k-1}\left(u_\varepsilon\right)\right) \\
= &\int_{Q_T\cap\{k-1<u_\varepsilon\leq k\}} f_\varepsilon T_1\left(u_\varepsilon - T_{k-1}\left(u_\varepsilon\right)\right) + \int_{Q_T\cap\{u_\varepsilon>k\}} f_\varepsilon \\
= &\int_{Q_T\cap\{u_\varepsilon\leq k-1\}} f + \int_{Q_T\cap\{k-1<u_\varepsilon\leq k\}} f + \int_{Q_T\cap\{u_\varepsilon>k\}} f
\end{aligned}$$

also we have

$$\int_\Omega S_1\left(u_0\right) = \int_\Omega \int_0^{u_0} T_1\left(s - T_{k-1}\left(s\right)\right) ds = \int_\Omega \int_{[0,u_0]\cap\{s>k-1\}} T_1\left(s - T_{k-1}\left(s\right)\right) ds$$

Therefore, from (9) combined with the two later inequalities and the above equality, we obtain

$$\begin{aligned}
&\int_{E\cap\{u_\varepsilon>k\}} d\left(t,x\right) \frac{u_\varepsilon\left|\nabla u_\varepsilon\right|^p}{\left(u_\varepsilon+\varepsilon\right)^{\gamma+1}} \\
\leq &\int_{Q\cap\{u_\varepsilon\geq k-1\}} f + \int_E \int_{[0,u_0]\cap\{s>k-1\}} T_1\left(s - T_{k-1}\left(s\right)\right) ds
\end{aligned}$$

It follows from $f \in L^m\left(Q\right)$ and $T_1\left(s - T_{k-1}\left(s\right)\right) \in L^1\left(\Omega\right)$ that

$$\int_{E\cap\{u_\varepsilon>k\}} d\left(t,x\right) \frac{u_\varepsilon\left|\nabla u_\varepsilon\right|^p}{\left(u_\varepsilon+\varepsilon\right)^{\gamma+1}} \to 0 \text{ as } k\to\infty$$

Then, there exists $k_0 > 1$ such that

$$\int_{E\cap\{u_\varepsilon>k\}} d\left(t,x\right) \frac{u_\varepsilon\left|\nabla u_\varepsilon\right|^p}{\left(u_\varepsilon+\varepsilon\right)^{\gamma+1}} \leq \frac{\nu}{2},\ \forall k>k_0,\ \forall\varepsilon\in(0,T) \qquad (10)$$

Since from (**??**) $\left(T_k\left(u_\varepsilon\right)\to T_k\left(u\right)\right)$ strongly in $L^p\left(0,T;W_0^{1,p}\left(\Omega\right)\right)$, then there exists $\varepsilon_\nu,\ \theta_\nu$ such that $|E|\leq\theta_\nu$, and we have

$$\frac{1}{c_\varepsilon^\gamma}\int_E d\left(t,x\right)\left|\nabla T_k\left(u_\varepsilon\right)\right|^p \leq \frac{\nu}{2},\ \forall\varepsilon\leq\varepsilon_\nu \qquad (11)$$

The estimates (10) and (11) imply that $d\left(t,x\right)\frac{u_{\varepsilon}\left|\nabla u_{\varepsilon}\right|^{p}}{\left(u_{\varepsilon}+\varepsilon\right)^{\gamma+1}}$ is equi-integrable. This fact, together with the a.e. convergence of this term to $d\left(t,x\right)\frac{\left|\nabla u\right|^{p}}{u^{\gamma}}$, implies by the Vitali Theorem that

$$d\left(t,x\right)\frac{u_{\varepsilon}\left|\nabla u_{\varepsilon}\right|^{p}}{\left(u_{\varepsilon}+\varepsilon\right)^{\gamma+1}}\rightarrow d\left(t,x\right)\frac{\left|\nabla u\right|^{p}}{u^{\gamma}}\text{ , strongly locally in }L^{1}\left(Q_{T}\right)\qquad(12)$$

Let $\varphi\in C_{c}^{1}\left(\Omega\times[0,T)\right)$, taking φ test function in problem (4), by (5), (6), (8) and (12), we can let $\varepsilon\rightarrow 0$ yielding

$$\begin{aligned}&-\int_{Q_{T}}u\frac{\partial\varphi}{\partial t}+\int_{Q_{T}}a\left(t,x\right)\nabla u\nabla\varphi+\int_{Q_{T}}d\left(t,x\right)\frac{\left|\nabla u\right|^{p}}{u^{\gamma}}\varphi\\=&\int_{Q_{T}}f\varphi+\int_{\Omega}u_{0}\left(x\right)\varphi\left(x,0\right)\end{aligned}$$

Thus, Theorem 2 is proved. ■

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