

Nondifferentiable multiobjective programming under generalized d_I -invexity

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Résumé We are concerned with a nondifferentiable multiobjective programming problem with inequality constraints. We define new class of nondifferentiable functions called d_I -invexity in which each component of the objective function is directionally differentiable in its own direction d_i . In the following, from Suneja and Srivastava [6], Mishra and Noor [3], Slimani and Radjef [5], we introduce new classes of problems called d_I -V-type I, quasi-, pseudo-, pseudo quasi-, quasi pseudo- d_I -V-type I with respect to $(\eta_i)_i$ and $(\theta_j)_j$. In the framework of the new concepts, we prove new Fritz-John type necessary and Karush-Kuhn-Tucker type necessary and sufficient optimality conditions for a feasible point to be weakly efficient or efficient solution. Moreover, we prove weak and strong duality results for Mond-Weir type dual under various types of generalized d_I -V-type I requirements.

Keywords : Nondifferentiable multiobjective programming; Optimality; Duality; Directional derivative; d_I -invexity; d_I -V-type I; Weakly efficient, Efficient and Properly efficient solution.

4.1 Preliminaries and definitions

The following conventions for equalities and inequalities will be used. If $x, y \in \mathbb{R}^n$, then

$$x < y \Leftrightarrow x_i < y_i, \quad i=1, \dots, n;$$

$$x \leq y \Leftrightarrow x_i \leq y_i, \quad i=1, \dots, n;$$

$$x \leq y \Leftrightarrow x \leq y \text{ and } x \neq y.$$

We also note $\mathbb{R}_{\geq}^q$ (resp. $\mathbb{R}_{\geq}^q$ or $\mathbb{R}_{>}^q$) the set of vectors $y \in \mathbb{R}^q$ with $y \geq 0$ (resp. $y \geq 0$ or $y > 0$).

Definition 1. [2] If $f : D \rightarrow \mathbb{R}$ is locally Lipschitz at $x_0 \in D$, the directional derivative (in the sense of Clarke) of f at x_0 in the direction $d \in \mathbb{R}^n$, denoted by $f'(x_0; d)$, is given by

$$f'(x_0; d) = \lim_{\lambda \rightarrow 0^+} \frac{f(x_0 + \lambda d) - f(x_0)}{\lambda},$$

whenever this limit exists.

f is directionally differentiable at x_0 if its directional derivative exists finite in all directions.

Ye [7] has defined the invexity in the nondifferentiable case for the function f by taking all its components directionally differentiable in the same direction d . In the following definition, we define the d_I -invexity at a point x_0 for f , where each component f_i , $i = 1, \dots, N$ is directionally differentiable in its own direction d_i .

Definition 2. Let $f : D \rightarrow \mathbb{R}^N$ be defined on a non-empty open set $D \subset \mathbb{R}^n$ and for all $i = 1, \dots, N$, f_i is directionally differentiable at $x_0 \in D$ (not necessary in all directions). f is called d_I -invex at x_0 on D with respect to $(\eta_i)_{i=1, \dots, N}$, if there exists N vector functions $\eta_i : D \times D \rightarrow \mathbb{R}^n$, $i = 1, \dots, N$, such that for any $x \in D$

$$f_i(x) - f_i(x_0) \geq f'_i(x_0; \eta_i(x, x_0)), \text{ for all } i = 1, \dots, N, \quad (4.1)$$

where $f'_i(x_0; \eta_i(x, x_0))$ denotes the directional derivative of f_i in the direction $\eta_i(x, x_0)$.

If inequalities (7.2) are satisfied at any point $x_0 \in D$, then f is said to be d_I -invex on D with respect to $(\eta_i)_{i=1, \dots, N}$.

Remark 1. In the above definition, it doesn't require that the directional derivative exists finite in all directions for each component f_i , $i = 1, \dots, N$. Further, it is not necessary that all the components to be directionally differentiable in the same direction, which may widen the area of application.

In order to illustrate the definition 2, we give the following example.

Example 1. Consider the function $f = (f_1, f_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$f_1(x_1, x_2) = \begin{cases} 0, & \text{if } x_1 = 0 \text{ or } x_2 = 0; \\ 1 + x_1^4, & \text{otherwise.} \end{cases}$$

$$f_2(x_1, x_2) = \begin{cases} 1 + x_1^2, & \text{if } x_1 \neq 0 \text{ and } x_2 = 0; \\ 1 + x_2^2, & \text{if } x_1 = 0 \text{ and } x_2 \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, f_1 and f_2 are not differentiable functions at $x_0 = (0, 0)$, but only directionally differentiable functions at this point. Further there exists no function $\eta(x, x_0) \neq (0, 0)$ such that for any $x \in \mathbb{R}^2$, $f'_1(x_0; \eta(x, x_0))$ and $f'_2(x_0; \eta(x, x_0))$ exist and finite, it follows that there exists no function $\eta(x, x_0) \neq (0, 0)$ for which f is d-invex at x_0 on $\mathbb{R}^n$. But f_1 is directionally differentiable at x_0 in the direction $\eta_1(x, x_0) = (0, \gamma(x, x_0))$ (or $\eta_1(x, x_0) = (\gamma(x, x_0), 0)$) with $\gamma(x, x_0) \neq 0$, $\forall x \in \mathbb{R}^2$, f_2 is directionally differentiable at x_0 in the direction $\eta_2(x, x_0) = (\alpha(x, x_0), \beta(x, x_0))$ with $\alpha(x, x_0) \neq 0$, $\beta(x, x_0) \neq 0$, $\forall x \in \mathbb{R}^2$ and f is d_I -invex at x_0 on $\mathbb{R}^n$ with respect to $(\eta_i)_{i=1, 2}$.

Consider the following multiobjective programming problem

$$(VP) \quad \begin{aligned} &\text{Minimize } f(x) = (f_1(x), \dots, f_N(x)), \\ &\text{subject to } g(x) \leq 0, \end{aligned}$$

where $f : D \rightarrow \mathbb{R}^N$, $g : D \rightarrow \mathbb{R}^k$, D is a nonempty open subset of $\mathbb{R}^n$.

Let $X = \{x \in D / g(x) \leq 0\}$, the set of feasible solutions of (VP) and for $x_0 \in X$, we denote $J(x_0) = \{j \in \{1, \dots, k\} / g_j(x_0) = 0\}$, $\tilde{J}(x_0) = \{j \in \{1, \dots, k\} / g_j(x_0) < 0\}$;

Now following Suneja and Srivastava [6], Mishra and Noor [3], Slimani and Radjef [5] and the definition 2, we define the generalized d -V-type I problems.

Definition 3. We say that the problem (VP) is of d_I -V-type I at $x_0 \in X$ with respect to $(\eta_i)_{i=\overline{1,N}}$ and $(\theta_j)_{j=\overline{1,k}}$, if there exists $(N+k)$ vector functions $\eta_i : X \times X \rightarrow \mathbb{R}^n$, $i = \overline{1,N}$ and $\theta_j : X \times X \rightarrow \mathbb{R}^n$, $j = \overline{1,k}$ such that for all $x \in X$:

$$f_i(x) - f_i(x_0) \geq f'_i(x_0; \eta_i(x, x_0)), \quad \forall i = 1, \dots, N, \quad (4.2)$$

$$-g_j(x_0) \geq g'_j(x_0; \theta_j(x, x_0)), \quad \forall j = 1, \dots, k. \quad (4.3)$$

Definition 4. We say that the problem (VP) is of quasi d_I -V-type I at $x_0 \in X$ with respect to $(\eta_i)_{i=\overline{1,N}}$ and $(\theta_j)_{j=\overline{1,k}}$, if there exists $(N+k)$ vector functions $\eta_i : X \times X \rightarrow \mathbb{R}^n$, $i = \overline{1,N}$ and $\theta_j : X \times X \rightarrow \mathbb{R}^n$, $j = \overline{1,k}$ such that for some vectors $\mu \in \mathbb{R}_{\geq}^N$ and $\lambda \in \mathbb{R}_{\geq}^k$:

$$\sum_{i=1}^N \mu_i [f_i(x) - f_i(x_0)] \leq 0 \quad \Rightarrow \quad \sum_{i=1}^N \mu_i f'_i(x_0; \eta_i(x, x_0)) \leq 0, \quad \forall x \in X, \quad (4.4)$$

$$\sum_{j=1}^k \lambda_j g_j(x_0) \geq 0 \quad \Rightarrow \quad \sum_{j=1}^k \lambda_j g'_j(x_0; \theta_j(x, x_0)) \leq 0, \quad \forall x \in X. \quad (4.5)$$

Definition 5. We say that the problem (VP) is of pseudo d_I -V-type I at $x_0 \in X$ with respect to $(\eta_i)_{i=\overline{1,N}}$ and $(\theta_j)_{j=\overline{1,k}}$, if there exists $(N+k)$ vector functions $\eta_i : X \times X \rightarrow \mathbb{R}^n$, $i = \overline{1,N}$ and $\theta_j : X \times X \rightarrow \mathbb{R}^n$, $j = \overline{1,k}$ such that for some vectors $\mu \in \mathbb{R}_{\geq}^N$ and $\lambda \in \mathbb{R}_{\geq}^k$:

$$\sum_{i=1}^N \mu_i f'_i(x_0; \eta_i(x, x_0)) \geq 0 \quad \Rightarrow \quad \sum_{i=1}^N \mu_i [f_i(x) - f_i(x_0)] \geq 0, \quad \forall x \in X, \quad (4.6)$$

$$\sum_{j=1}^k \lambda_j g'_j(x_0; \theta_j(x, x_0)) \geq 0 \quad \Rightarrow \quad \sum_{j=1}^k \lambda_j g_j(x_0) \leq 0, \quad \forall x \in X. \quad (4.7)$$

Definition 6. We say that the problem (VP) is of quasi pseudo d_I -V-type I at $x_0 \in X$ with respect to $(\eta_i)_{i=\overline{1,N}}$ and $(\theta_j)_{j=\overline{1,k}}$, if there exists $(N+k)$ vector functions $\eta_i : X \times X \rightarrow$

$\mathbb{R}^n, i = \overline{1, N}$ and $\theta_j : X \times X \rightarrow \mathbb{R}^n, j = \overline{1, k}$ such that for some vectors $\mu \in \mathbb{R}_{\geq}^N$ and $\lambda \in \mathbb{R}_{\geq}^k$:

$$\sum_{i=1}^N \mu_i [f_i(x) - f_i(x_0)] \leq 0 \Rightarrow \sum_{i=1}^N \mu_i f'_i(x_0; \eta_i(x, x_0)) \leq 0, \forall x \in X, \quad (4.8)$$

$$\sum_{j=1}^k \lambda_j g'_j(x_0; \theta_j(x, x_0)) \geq 0 \Rightarrow \sum_{j=1}^k \lambda_j g_j(x_0) \leq 0, \forall x \in X. \quad (4.9)$$

Definition 7. We say that the problem (VP) is of pseudo quasi d_I -V-type I at $x_0 \in X$ with respect to $(\eta_i)_{i=\overline{1, N}}$ and $(\theta_j)_{j=\overline{1, k}}$, if there exists $(N+k)$ vector functions $\eta_i : X \times X \rightarrow \mathbb{R}^n, i = \overline{1, N}$ and $\theta_j : X \times X \rightarrow \mathbb{R}^n, j = \overline{1, k}$ such that for some vectors $\mu \in \mathbb{R}_{\geq}^N$ and $\lambda \in \mathbb{R}_{\geq}^k$:

$$\sum_{i=1}^N \mu_i f'_i(x_0; \eta_i(x, x_0)) \geq 0 \Rightarrow \sum_{i=1}^N \mu_i [f_i(x) - f_i(x_0)] \geq 0, \forall x \in X, \quad (4.10)$$

$$\sum_{j=1}^k \lambda_j g_j(x_0) \geq 0 \Rightarrow \sum_{j=1}^k \lambda_j g'_j(x_0; \theta_j(x, x_0)) \leq 0, \forall x \in X. \quad (4.11)$$

4.2 Optimality conditions

In recent paper, Slimani and Radjef [5] considered a number of necessary and sufficient optimality conditions which depend on generalized V-type I problems with respect to $(\eta_i)_{i=\overline{1, N}}$ and $(\theta_j)_{j=\overline{1, k}}$. In this section, we adapt their results to the classes of nondifferentiable problems.

Theorem 1. (Sufficiency) Let x_0 be a feasible solution for (VP) and suppose that there exists $(N+J)$ vector functions $\eta_i : X \times X \rightarrow \mathbb{R}^n, i = \overline{1, N}, \theta_j : X \times X \rightarrow \mathbb{R}^n, j \in J(x_0)$ and scalars $\mu_i \geq 0, i = \overline{1, N}, \sum_{i=1}^N \mu_i = 1, \lambda_j \geq 0, j \in J(x_0)$ such that the following relation is satisfied :

$$\sum_{i=1}^N \mu_i f'_i(x_0; \eta_i(x, x_0)) + \sum_{j \in J(x_0)} \lambda_j g'_j(x_0; \theta_j(x, x_0)) \geq 0, \forall x \in X. \quad (4.12)$$

Moreover, assume that one of the following conditions is satisfied :

- (a) the problem (VP) is quasi strictly-pseudo d_I -V-type I at x_0 with respect to $(\eta_i)_{i=\overline{1, N}}$ and $(\theta_j)_{j \in J(x_0)}$;

- (b) the problem (VP) is semi strictly-quasi d_I -V-type I at x_0 with respect to $(\eta_i)_{i=\overline{1,N}}$, $(\theta_j)_{j \in J(x_0)}$ and for μ and λ ;
- (c) the problem (VP) is strictly-pseudo d_I -V-type I at x_0 with respect to $(\eta_i)_{i=\overline{1,N}}$, $(\theta_j)_{j \in J(x_0)}$ and for μ and λ .

Then x_0 is an efficient solution for (VP).

Antczak [1] has given necessary conditions for $x_0 \in X$ to be weakly efficient solution for (VP) by taking the functions f_i , $i = \overline{1,N}$ and g_j , $j \in J(x_0)$ directionally differentiable in the same direction $\eta(x, x_0)$. Now we give necessary optimality criteria by considering each function f_i , $i = \overline{1,N}$ (resp. g_j , $j \in J(x_0)$) directionally differentiable in its own direction $\eta_i(x, x_0)$, $i = \overline{1,N}$ (resp. $\theta_j(x, x_0)$, $j \in J(x_0)$).

Theorem 2. (Fritz John type necessary optimality conditions) Suppose that

- (i) x_0 is a weakly efficient solution for (VP);
- (ii) g_j is continuous at x_0 for $j \in \tilde{J}(x_0)$ and there exists vector functions $\eta_i : X \times X \rightarrow \mathbb{R}^n$, $i = \overline{1,N}$, $\theta_j : X \times X \rightarrow \mathbb{R}^n$, $j \in J(x_0)$ which satisfy at x_0 with respect to $\eta : X \times X \rightarrow \mathbb{R}^n$ the following inequalities for all $x \in X$,

$$f'_i(x_0; \eta(x, x_0)) \leq f'_i(x_0; \eta_i(x, x_0)), \quad i = \overline{1,N}, \quad (4.13)$$

$$g'_j(x_0; \eta(x, x_0)) \leq g'_j(x_0; \theta_j(x, x_0)), \quad j \in J(x_0); \quad (4.14)$$

- (iii) the functions $f'_i(x_0; \eta_i(x, x_0))$, $i = \overline{1,N}$ and $g'_j(x_0; \theta_j(x, x_0))$, $j \in J(x_0)$ are pre-invex of x on X with respect to $\gamma : X \times X \rightarrow \mathbb{R}^n$;

Then there exists $(\mu, \lambda) \in \mathbb{R}_{\geq}^{N+|J(x_0)|}$ such that (x_0, μ, λ) satisfies

$$\sum_{i=1}^N \mu_i f'_i(x_0; \eta_i(x, x_0)) + \sum_{j \in J(x_0)} \lambda_j g'_j(x_0; \theta_j(x, x_0)) \geq 0, \quad \forall x \in X. \quad (4.15)$$

Now, we need a constraint qualification to prove the next result.

Definition 8. The nondifferentiable function g is said to satisfy the d_I -constraint qualification at $x_0 \in X$ with respect to $(\theta_j)_{j \in J(x_0)}$ if there exists $\bar{x} \in X$ and $\theta_j : X \times X \rightarrow \mathbb{R}^n$, $j \in J(x_0)$ such that

$$g'_j(x_0; \theta_j(\bar{x}, x_0)) < 0, \quad \forall j \in J(x_0). \quad (4.16)$$

Thus, the following KKT type necessary optimality conditions for (VP) are satisfied.

Theorem 3. (Karush-Kuhn-Tucker type necessary optimality conditions) Suppose that the conditions (i), (ii) and (iii) of Theorem 6.1 are satisfied and the function g satisfies the d_I -constraint qualification at $x_0 \in X$ with respect to $(\theta_j)_{j \in J(x_0)}$. Then there exists $\mu \in \mathbb{R}_{\geq}^N$ and $\lambda \in \mathbb{R}_{\geq}^{|J(x_0)|}$ such that (x_0, μ, λ) satisfies the relation (4.15).

4.3 Mond-Weir duality

In relation to (VP) and using the relation (4.15), we consider the following multiobjective dual problem, which is in the format of Mond-Weir [4].

$$(MWD) \quad \text{Maximize } f(y) = (f_1(y), \dots, f_N(y)),$$

subject to

$$\sum_{i=1}^N \mu_i f'_i(y; \eta_i(x, y)) + \sum_{j \in J(y)} \lambda_j g'_j(y; \theta_j(x, y)) \geq 0, \quad \forall x \in X. \quad (4.17)$$

$y \in X$, $\mu \in \mathbb{R}_{\geq}^N$, $\lambda \in \mathbb{R}_{\geq}^{|J(y)|}$, $\eta_i : X \times X \rightarrow \mathbb{R}^n$, $\forall i = 1, \dots, N$, $\theta_j : X \times X \rightarrow \mathbb{R}^n$, $\forall j \in J(y)$.

We let Y be the set of feasible solutions of problem (MWD);

We denote by $Pr_X Y$ the projection of set Y on X .

Theorem 4. (*Weak duality*) Let x_0 and $(y, \mu, \lambda, (\eta_i)_{i=\overline{1, N}}, (\theta_j)_{j \in J(y)})$ be feasible solutions for (VP) and (MWD) respectively. Moreover, assume that one of the following conditions is satisfied :

- (a) the problem (VP) is pseudo quasi d_I -V-type I at y with respect to $(\eta_i)_{i=\overline{1, N}}, (\theta_j)_{j \in J(y)}$ and for μ and λ (with $\mu > 0$);
- (b) the problem (VP) is strictly-pseudo quasi d_I -V-type I at y with respect to $(\eta_i)_{i=\overline{1, N}}, (\theta_j)_{j \in J(y)}$ and for μ and λ .
- (c) the problem (VP) is quasi strictly-pseudo d_I -V-type I at y with respect to $(\eta_i)_{i=\overline{1, N}}, (\theta_j)_{j \in J(y)}$ and for μ and λ .

Then $f(x) \not\leq f(y)$.

Remark 2. If we omit the assumption $\mu > 0$ in the condition (a) or the word strictly in the condition (b), we obtain, for this part of theorem, $f(x) \not\leq f(y)$.

Theorem 5. (*Strong duality*) Let x_0 be weakly efficient solution for (VP) and suppose that the conditions (ii) and (iii) of Theorem 6.1 are satisfied. Assume also that the function g satisfies the d_I -constraint qualification at x_0 with respect to $(\theta_j)_{j \in J(x_0)}$. Then there exists $\mu \in \mathbb{R}_{\geq}^N$ and $\lambda \in \mathbb{R}_{\geq}^{|J(x_0)|}$ such that $(x_0, \mu, \lambda, (\eta_i)_{i=\overline{1, N}}, (\theta_j)_{j \in J(x_0)}) \in Y$ and the objective functions of (VP) and (MWD) have the same values at x_0 and $(x_0, \mu, \lambda, (\eta_i)_{i=\overline{1, N}}, (\theta_j)_{j \in J(x_0)})$, respectively. If, further, the weak duality with the condition (a) without $(\mu > 0)$ (resp. with the condition (b) or (c)) between (VP) and (MWD) in theorem 4 holds, then $(x_0, \mu, \lambda, (\eta_i)_{i=\overline{1, N}}, (\theta_j)_{j \in J(x_0)}) \in Y$ is a weakly efficient (resp. an efficient) solution of (MWD).

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