

Duality for nonlinear programming under invexity with respect to different η_i

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Résumé In [6, 8], Radjef and Slimani are considered the invexity and its extensions for objective and constraint functions with respect to different η_i . Optimality conditions, for nonlinear and multiobjective programming problems, are obtained under this generalized invexity. In [7], for multiobjective programming, some duality results are obtained for a dual in the format of Mond-Weir under generalized V-type I invexity with respect to different $(\eta_i)_i$ and $(\theta_j)_j$. Further, in [6, 7, 8], a generalized Kuhn-Tucker relation is introduced which is necessary and sufficient for a feasible point to be optimal under various types of generalized invexity requirements. Numerical examples are constructed and show that the results obtained in [6, 7, 8] are applicable to prove that a feasible point is optimal when this point is not Kuhn-Tucker stationary (resp. a vector Kuhn-Tucker) for a nonlinear (resp. multiobjective) programming problem. In this paper, following the work given in [6] and using the generalized Kuhn-Tucker relation, two dual programs in the format of Wolfe and Mond-Weir are considered. Weak, strong, converse and strict duality results are obtained, for each dual programme, under the defined generalized invexity assumptions.

Mots clés : Nonlinear programming; KT-invexity with respect to η and $(\theta_j)_j$; KT-pseudo-invexity with respect to η and $(\theta_j)_j$; Generalized Kuhn-Tucker relation; Weak, strong, converse and strict duality; Optimal solution.

In [6, 8], Radjef and Slimani are considered the invexity and its extensions for objective and constraint functions with respect to different η_i . Optimality conditions, for nonlinear and multiobjective programming problems, are obtained under this generalized invexity. In [7], for multiobjective programming, some duality results are obtained for a dual in the format of Mond-Weir under generalized V-type I invexity with respect to different $(\eta_i)_i$ and $(\theta_j)_j$. Further, in [6, 7, 8], a generalized Kuhn-Tucker relation is introduced which is necessary and sufficient for a feasible point to be optimal under various types of generalized invexity requirements. Numerical examples are constructed and show that the results obtained in [6, 7, 8] are applicable to prove that a feasible point is optimal when this point is not Kuhn-Tucker stationary (resp. a vector Kuhn-Tucker) for a nonlinear (resp. multiobjective) programming problem. In this paper, following the work given in [6] and using the generalized Kuhn-Tucker relation, two dual programs in the format of Wolfe and

Mond-Weir are considered. Weak, strong, converse and strict duality results are obtained, for each dual programme, under the defined generalized invexity assumptions.

3.1 Preliminaries and definitions

We note $\mathbb{R}_{\geq}^q$ the set of vectors $y \in \mathbb{R}^q$ with $y \geq 0_{(q)}$, where for all $x, y \in \mathbb{R}^n$, $x \geq y \Leftrightarrow x_i \geq y_i, i = 1, \dots, n$.

Definition 1. [2] Let $f : D \rightarrow \mathbb{R}$ be defined on a non-empty set $D \subset \mathbb{R}^n$. The function f is pre-invex (with respect to η) on D if there exists a vector function $\eta : D \times D \rightarrow \mathbb{R}^n$ such that the inequality

$$f(x_0 + \lambda\eta(x, x_0)) \leq \lambda f(x) + (1 - \lambda)f(x_0), \quad (3.1)$$

holds for all $x, x_0 \in D$ and any $\lambda \in [0, 1]$.

Consider the following constrained nonlinear programming problem (P) :

$$(P) \quad \begin{array}{ll} \text{Minimize} & f(x), \\ \text{subject to} & g(x) \leq 0, \end{array}$$

where $f : D \rightarrow \mathbb{R}$ and $g_j : D \rightarrow \mathbb{R}, j = 1, \dots, k$ are differentiable functions with D is an open set of $\mathbb{R}^n$, $X = \{x \in D \subseteq \mathbb{R}^n / g_j(x) \leq 0, j = 1, \dots, k\}$ is the set of feasible solutions for (P). For $x_0 \in X$, we denote $J(x_0) = \{j \in \{1, \dots, k\} / g_j(x_0) = 0\}$, and $\tilde{J}(x_0) = \{j \in \{1, \dots, k\} / g_j(x_0) < 0\}$. For all $x \in D$, $g(x) = (g_1(x), \dots, g_k(x))^t$.

Now we introduce new class of problems by generalizing the class of KT-invex problem with respect to η defined by Martin [5].

Definition 2. The problem (P) is said to be KT-invex at $x_0 \in X$ with respect to η and $(\theta_j)_{j \in J(x_0)}$, if there exists $\eta : X \times X \rightarrow \mathbb{R}^n$ and $\theta_j : X \times X \rightarrow \mathbb{R}^n, j \in J(x_0)$ such that for each $x \in X$:

$$f(x) - f(x_0) \geq [\nabla f(x_0)]^t \eta(x, x_0), \quad (3.2)$$

$$-[\nabla g_j(x_0)]^t \theta_j(x, x_0) \geq 0, \quad \forall j \in J(x_0). \quad (3.3)$$

The problem (P) is said to be KT-invex on X with respect to η and $(\theta_j)_{j \in J(x_0)}$, if it is KT-invex at each $x_0 \in X$ with respect to the same η and $(\theta_j)_{j \in J(x_0)}$. If the second (implied) inequality in (3.2) is strict ($x \neq x_0$), we say that (P) is semi strictly KT-invex at x_0 (or on X as the case may be) with respect to η and $(\theta_j)_{j \in J(x_0)}$.

If the relation (3.3) is verified for $j = 1, \dots, k$, we say that the problem (P) is (semi strictly) KT-invex at x_0 (or on X) with respect to η and $(\theta_j)_{j=1, \dots, k}$.

Definition 3. The problem (P) is said to be *KT-pseudo-invex* at $x_0 \in X$ with respect to η and $(\theta_j)_{j \in J(x_0)}$, if there exists $\eta : X \times X \rightarrow \mathbb{R}^n$ and $\theta_j : X \times X \rightarrow \mathbb{R}^n$, $j \in J(x_0)$ such that for each $x \in X$:

$$[\nabla f(x_0)]^t \eta(x, x_0) \geq 0 \Rightarrow f(x) - f(x_0) \geq 0, \quad (3.4)$$

$$-[\nabla g_j(x_0)]^t \theta_j(x, x_0) \geq 0, \quad \forall j \in J(x_0). \quad (3.5)$$

The problem (P) is said to be *KT-pseudo-invex on X* with respect to η and $(\theta_j)_{j \in J(x_0)}$, if it is *KT-pseudo-invex* at each $x_0 \in X$ with respect to the same η and $(\theta_j)_{j \in J(x_0)}$. If the second (implied) inequality in (3.4) is strict ($x \neq x_0$), we say that (P) is *semi strictly KT-pseudo-invex* at x_0 (or on X as the case may be) with respect to η and $(\theta_j)_{j \in J(x_0)}$. If the relation (3.5) is verified for $j = 1, \dots, k$, we say that the problem (P) is (semi strictly) *KT-pseudo-invex* at x_0 (or on X) with respect to η and $(\theta_j)_{j=1, \dots, k}$.

Theorem 1. (necessity) Suppose that

- (i) x_0 is a local or a global solution for (P);
- (ii) g_j is continuous at x_0 for $j \in \tilde{J}(x_0)$ and there exists vector functions $\eta : X \times X \rightarrow \mathbb{R}^n$, $\theta_j : X \times X \rightarrow \mathbb{R}^n$, $j \in J(x_0)$ which satisfy at x_0 with respect to $\phi : X \times X \rightarrow \mathbb{R}^n$ the following inequalities for all $x \in X$,

$$[\nabla f(x_0)]^t \phi(x, x_0) \leq [\nabla f(x_0)]^t \eta(x, x_0), \quad (3.6)$$

$$[\nabla g_j(x_0)]^t \phi(x, x_0) \leq [\nabla g_j(x_0)]^t \theta_j(x, x_0), \quad j \in J(x_0). \quad (3.7)$$

- (iii) the functions $[\nabla f(x_0)]^t \eta(x, x_0)$ and $[\nabla g_j(x_0)]^t \theta_j(x, x_0)$, $j \in J(x_0)$ are pre-invex of x on X with respect to $\gamma : X \times X \rightarrow \mathbb{R}^n$;
- (iv) there exists $\bar{x} \in X$ such that

$$-g_j(x_0) > [\nabla g_j(x_0)]^t \theta_j(\bar{x}, x_0), \quad \forall j \in J(x_0). \quad (3.8)$$

Then there exists $\lambda \in \mathbb{R}_{\geq}^{|J(x_0)|}$ such that (x_0, λ) satisfies the following generalized Kuhn-Tucker relation

$$[\nabla f(x_0)]^t \eta(x, x_0) + \sum_{j \in J(x_0)} \lambda_j [\nabla g_j(x_0)]^t \theta_j(x, x_0) \geq 0, \quad \forall x \in X. \quad (3.9)$$

3.2 Wolfe Duality

In relation to (P) we consider the following dual problem, which is in the format of Wolfe.

$$(WD) \quad \text{Maximize } f(y) + \lambda^t g(y),$$

subject to

$$[\nabla f(y)]^t \eta(x, y) + \sum_{j=1}^k \lambda_j [\nabla g_j(y)]^t \theta_j(x, y) \geq 0, \quad \forall x \in X, \quad (3.10)$$

$$\lambda \in \mathbb{R}_{\geq}^k, \quad \eta : X \times X \rightarrow \mathbb{R}^n,$$

$$\theta_j : X \times X \rightarrow \mathbb{R}^n, \quad \forall j = 1, \dots, k.$$

Let $Y = \left\{ (y, \lambda, \eta, (\theta_j)_j) : [\nabla f(y)]^t \eta(x, y) + \sum_{j=1}^k \lambda_j [\nabla g_j(y)]^t \theta_j(x, y) \geq 0, \quad \forall x \in X; \right.$
 $\left. \lambda \in \mathbb{R}_{\geq}^k; \eta : X \times X \rightarrow \mathbb{R}^n, \theta_j : X \times X \rightarrow \mathbb{R}^n, \quad \forall j = 1, \dots, k \right\}$ the set of feasible solutions of problem (WD).

We denote by $Pr_X Y = \{y \in X : (y, \lambda, \eta, (\theta_j)_{j=\overline{1,k}}) \in Y\}$.

It is known that weak and strong duality in the sense of Wolfe hold in the case when the functions occurring in problem (Wolfe Dual), are convex [1, 4] or invex with respect to the same function η [3].

Now we establish certain duality results between (P) and (WD) by using the KT-invexity notion with respect to different η and $(\theta_j)_j$.

Theorem 2. (*Weak duality*) Suppose that

i) $x \in X$;

ii) $(y, \lambda, \eta, (\theta_j)_{j=\overline{1,k}}) \in Y$;

iii) the problem (P) is KT-invex at y with respect to η and $(\theta_j)_{j=\overline{1,k}}$.

Then $f(x) \not\leq f(y) + \lambda^t g(y)$.

Theorem 3. (*Strong duality*) Suppose that

1) x_0 is an optimal solution for (P);

2) the hypothesis (ii), (iii) and (iv) of theorem 1 are satisfied.

Then there exists $\lambda \in \mathbb{R}_{\geq}^k$ such that $(x_0, \lambda, \eta, (\theta_j)_{j=\overline{1,k}}) \in Y$ and the objective functions of (P) and (WD) have the same values at x_0 and $(x_0, \lambda, \eta, (\theta_j)_{j=\overline{1,k}})$, respectively. If, further, the problem (P) is KT-invex at any $\bar{y} \in Pr_X Y$ with respect to $\bar{\eta}$ and $(\bar{\theta}_j)_{j=\overline{1,k}}$ (with $(\bar{y}, \bar{\lambda}, \bar{\eta}, (\bar{\theta}_j)_{j=\overline{1,k}}) \in Y$), then $(x_0, \lambda, \eta, (\theta_j)_{j=\overline{1,k}}) \in Y$ is an optimal solution of (WD).

Theorem 4. (*Converse duality*) Suppose that

(i) $(y, \lambda, \eta, (\theta_j)_{j=\overline{1,k}}) \in Y$;

(ii) $y \in X$;

(iii) the problem (P) is KT-pseudo-invex at y with respect to η and $(\theta_j)_{j=\overline{1,k}}$.

Then y is an optimal solution for (P) .

Theorem 5. (Strict duality) Suppose that

i) $x^0 \in X$ and $(y^0, \lambda^0, \eta, (\theta_j)_{j=\overline{1,k}}) \in Y$ such that

$$f(x^0) = f(y^0) + \lambda^{0t} g(y^0); \quad (3.11)$$

ii) the problem (P) is semi strictly KT-invex at y^0 with respect to η and $(\theta_j)_{j=\overline{1,k}}$.
Then $x^0 = y^0$.

3.3 Mond-Weir Duality

In relation to (P) we consider the following dual problem, which is in the format of Mond-Weir.

$$(MWD) \quad \text{Maximize } f(y),$$

subject to

$$[\nabla f(y)]^t \eta(x, y) + \sum_{j \in J(y)} \lambda_j [\nabla g_j(y)]^t \theta_j(x, y) \geq 0, \quad \forall x \in X, \quad (3.12)$$

$$\lambda \in \mathbb{R}_{\geq}^{|J(y)|}, \quad \eta : X \times X \rightarrow \mathbb{R}^n,$$

$$\theta_j : X \times X \rightarrow \mathbb{R}^n, \quad \forall j \in J(y).$$

Let $\bar{Y} = \left\{ (y, \lambda, \eta, (\theta_j)_j) : [\nabla f(y)]^t \eta(x, y) + \sum_{j \in J(y)} \lambda_j [\nabla g_j(y)]^t \theta_j(x, y) \geq 0, \quad \forall x \in X; \right.$
 $\left. \lambda \in \mathbb{R}_{\geq}^{|J(y)|}; \quad \eta : X \times X \rightarrow \mathbb{R}^n, \quad \theta_j : X \times X \rightarrow \mathbb{R}^n, \quad \forall j \in J(y) \right\}$ the set of feasible solutions of problem (MWD).

We denote by $Pr_X \bar{Y} = \{y \in X : (y, \lambda, \eta, (\theta_j)_{j \in J(y)}) \in \bar{Y}\}$.

In the following, we establish some duality results for the problem (MWD) while using the KT-pseudo-invexity notion with respect to different η and $(\theta_j)_j$.

Theorem 6. (Weak duality) Suppose that

i) $x \in X$;

ii) $(y, \lambda, \eta, (\theta_j)_{j \in J(y)}) \in \bar{Y}$;

iii) the problem (P) is KT-pseudo-invex at y with respect to η and $(\theta_j)_{j \in J(y)}$.

Then $f(x) \not\leq f(y)$.

Theorem 7. (Strong duality) Suppose that

1) x_0 is an optimal solution for (P) ;

2) the hypothesis (ii), (iii) and (iv) of theorem 1 are satisfied.

Then there exists $\lambda \in \mathbb{R}_{\geq}^{|J(x_0)|}$ such that $(x_0, \lambda, \eta, (\theta_j)_{j \in J(x_0)}) \in \bar{Y}$ and the objective functions of (P) and (MWD) have the same values at x_0 and $(x_0, \lambda, \eta, (\theta_j)_{j \in J(x_0)})$, respectively. If, further, the problem (P) is KT -pseudo-invex at any $\bar{y} \in Pr_X \bar{Y}$ with respect to $\bar{\eta}$ and $(\bar{\theta}_j)_{j \in J(y)}$ (with $(\bar{y}, \bar{\lambda}, \bar{\eta}, (\bar{\theta}_j)_{j \in J(x_0)}) \in \bar{Y}$), then $(x_0, \lambda, \eta, (\theta_j)_{j \in J(x_0)}) \in \bar{Y}$ is an optimal solution of (MWD) .

Theorem 8. (Converse duality) Suppose that

(i) $(y, \lambda, \eta, (\theta_j)_{j \in J(y)}) \in \bar{Y}$;

(ii) $y \in X$;

(iii) the problem (P) is KT -pseudo-invex at y with respect to η and $(\theta_j)_{j \in J(y)}$.

Then y is an optimal solution for (P) .

Theorem 9. (Strict duality) Suppose that

i) $x^0 \in X$ and $(y^0, \lambda^0, \eta, (\theta_j)_{j \in J(y^0)}) \in \bar{Y}$ such that

$$f(x^0) = f(y^0); \quad (3.13)$$

ii) the problem (P) is semi strictly KT -pseudo-invex at y^0 with respect to η and $(\theta_j)_{j \in J(y^0)}$.

Then $x^0 = y^0$.

3.4 Conclusion

In this paper, we have defined new classes of problem called KT -invex and KT -pseudo-invex with respect to η and $(\theta_j)_j$ as a generalization of KT -invex problem with respect to the same η defined by Martin [5]. In relation to the nonlinear programming, two dual programs in the format of Wolfe and Mond-Weir are considered involving the generalized Kuhn-Tucker relation. For each dual programme, weak, strong, converse and strict duality results are obtained under the defined generalized invexity.

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