

Generalized V-type I Invexity and Duality for Multiobjective programs

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Résumé As generalization of the invexity notion originated by Hanson [2], Hanson and Mond [3] have introduced type I and type II functions for studying non linear programming problems. Later, Kaul et al. [5], Aghezzaf and Hachimi [1] have defined the notion of type I function with respect to the same function η for constrained multiobjective problems. Hanson et al. [4] gave another definition of multiobjective V-type I problem with respect to η . In this paper, following the work of Radjef and Slimani [6], where optimality conditions have been obtained for a nonlinear programming program under invexity with respect to different η_i , we have introduced a, more general, notion of multiobjective V-type I problem with respect to functions $(\eta_i)_i$ and $(\theta_j)_j$ that are not necessarily the same. We have obtained necessary and sufficient conditions for a feasible point to be a weak efficient, efficient or properly efficient solution under various types of generalized V-type I requirements. We have defined a Mond-Weir type dual problem and we have proved weak, strong and converse duality theorems under the defined notions. Known results in the literature [4, 5] can be deduced as particular cases from our results, when the functions $(\eta_i)_i$ and $(\theta_j)_j$ are equal to the same function η which is different to zero.

Mots clés : Multiobjective problem, V-type I problem / $(\eta_i)_i$ and $(\theta_j)_j$; Quasi-, pseudo-, pseudo quasi-, quasi pseudo- V-type I / $(\eta_i)_i$ and $(\theta_j)_j$; Weak, strong and converse duality; Weak efficient, Efficient, Properly efficient solution.

As generalization of the invexity notion originated by Hanson [2], Hanson and Mond [3] have introduced type I and type II functions for studying non linear programming problems. Later, Kaul et al. [5], Aghezzaf and Hachimi [1] have defined the notion of type I function with respect to the same function η for constrained multiobjective problems. Hanson et al. [4] gave another definition of multiobjective V-type I problem with respect to η . In this paper, following the work of Radjef and Slimani [6], where optimality conditions have been obtained for a nonlinear programming program under invexity with respect to different η_i , we have introduced a, more general, notion of multiobjective V-type I problem with respect to functions $(\eta_i)_i$ and $(\theta_j)_j$ that are not necessarily the same. We have obtained necessary and sufficient conditions for a feasible point to be a weak efficient, efficient or properly efficient solution under various types of generalized V-type I requirements. We have defined a Mond-Weir type dual problem and

we have proved weak, strong and converse duality theorems under the defined notions. Known results in the literature [4, 5] can be deduced as particular cases from our results, when the functions $(\eta_i)_i$ and $(\theta_j)_j$ are equal to the same function η which is different to zero.

3.1 Preliminaries and definitions

The following conventions for inequalities will be used. If $x, y \in \mathbb{R}^n$, then

$$x \leq y \Leftrightarrow x_i \leq y_i, \quad i=1, \dots, n;$$

$$x \leq y \Leftrightarrow x \leq y \text{ and } x \neq y.$$

We also note $\mathbb{R}_{\geq}^q$ (resp. $\mathbb{R}_{\leq}^q$) the set of vectors $y \in \mathbb{R}^q$ with $y \geq 0_{(q)}$ (resp. $y \leq 0_{(q)}$).

We consider the following multiobjective optimization problem

$$(PV) \quad \begin{aligned} &\text{Minimize } f(x) = (f_1(x), \dots, f_N(x)), \\ &\text{subject to } g(x) \leq 0, \end{aligned}$$

where $f : D \rightarrow \mathbb{R}^N$ and $g : D \rightarrow \mathbb{R}^k$ are differentiable functions and $D \subseteq \mathbb{R}^n$ is an open set. $X = \{x \in D / g(x) \leq 0\}$. For $x_0 \in X$, $J(x_0) = \{j \in \{1, \dots, k\} / g_j(x_0) = 0\}$, $J = |J(x_0)|$.

In Radjef and Slimani [6], it has been constructed an example that shows the difficulty to find a function η with respect to which several other functions are invex. Consequently, in more general manner, the invexity notion was considered with respect to functions not necessarily the same. When we focused on this idea, and while inspiring of [1, 4, 5], we define the vector type I problem with respect to $(\eta_i)_{i=\overline{1, N}}$ and $(\theta_j)_{j=\overline{1, k}}$.

Définition 3.1 *We say the problem (PV) is of V-type I at $x_0 \in X$ with respect to $(\eta_i)_{i=\overline{1, N}}$ and $(\theta_j)_{j=\overline{1, k}}$, if there exists $N + k$ vector functions $\eta_i : X \times X \rightarrow \mathbb{R}^n$, $i = \overline{1, N}$ and $\theta_j : X \times X \rightarrow \mathbb{R}^n$, $j = \overline{1, k}$ such that for all $x \in X$:*

$$f_i(x) - f_i(x_0) \geq [\nabla f_i(x_0)]^t \eta_i(x, x_0), \quad \forall i = 1, \dots, N, \quad (3.1)$$

$$-g_j(x_0) \geq [\nabla g_j(x_0)]^t \theta_j(x, x_0), \quad \forall j = 1, \dots, k. \quad (3.2)$$

Définition 3.2 *We say the problem (PV) is of quasi V-type I at $x_0 \in X$ with respect to $(\eta_i)_{i=\overline{1, N}}$ and $(\theta_j)_{j=\overline{1, k}}$, if there exists $N + k$ vector functions $\eta_i : X \times X \rightarrow \mathbb{R}^n$, $i = \overline{1, N}$ and $\theta_j : X \times X \rightarrow \mathbb{R}^n$, $j = \overline{1, k}$ such that for some vectors $\mu \in \mathbb{R}_{\geq}^N$ and $\lambda \in \mathbb{R}_{\geq}^k$:*

$$\sum_{i=1}^N \mu_i [f_i(x) - f_i(x_0)] \leq 0 \quad \Rightarrow \quad \sum_{i=1}^N \mu_i [\nabla f_i(x_0)]^t \eta_i(x, x_0) \leq 0, \quad \forall x \in X, \quad (3.3)$$

$$\sum_{j=1}^k \lambda_j g_j(x_0) \geq 0 \quad \Rightarrow \quad \sum_{j=1}^k \lambda_j [\nabla g_j(x_0)]^t \theta_j(x, x_0) \leq 0, \quad \forall x \in X. \quad (3.4)$$

Définition 3.3 We say the problem (PV) is of pseudo V-type I at $x_0 \in X$ with respect to $(\eta_i)_{i=\overline{1,N}}$ and $(\theta_j)_{j=\overline{1,k}}$, if there exists $N + k$ vector functions $\eta_i : X \times X \rightarrow \mathbb{R}^n$, $i = \overline{1,N}$ and $\theta_j : X \times X \rightarrow \mathbb{R}^n$, $j = \overline{1,k}$ such that for some vectors $\mu \in \mathbb{R}_{\geq}^N$ and $\lambda \in \mathbb{R}_{\geq}^k$:

$$\sum_{i=1}^N \mu_i [\nabla f_i(x_0)]^t \eta_i(x, x_0) \geq 0 \Rightarrow \sum_{i=1}^N \mu_i [f_i(x) - f_i(x_0)] \geq 0, \forall x \in X, \quad (3.5)$$

$$\sum_{j=1}^k \lambda_j [\nabla g_j(x_0)]^t \theta_j(x, x_0) \geq 0 \Rightarrow \sum_{j=1}^k \lambda_j g_j(x_0) \leq 0, \forall x \in X. \quad (3.6)$$

Définition 3.4 We say the problem (PV) is of quasi pseudo V-type I at $x_0 \in X$ with respect to $(\eta_i)_{i=\overline{1,N}}$ and $(\theta_j)_{j=\overline{1,k}}$, if there exists $N + k$ vector functions $\eta_i : X \times X \rightarrow \mathbb{R}^n$, $i = \overline{1,N}$ and $\theta_j : X \times X \rightarrow \mathbb{R}^n$, $j = \overline{1,k}$ such that for some vectors $\mu \in \mathbb{R}_{\geq}^N$ and $\lambda \in \mathbb{R}_{\geq}^k$:

$$\sum_{i=1}^N \mu_i [f_i(x) - f_i(x_0)] \leq 0 \Rightarrow \sum_{i=1}^N \mu_i [\nabla f_i(x_0)]^t \eta_i(x, x_0) \leq 0, \forall x \in X, \quad (3.7)$$

$$\sum_{j=1}^k \lambda_j [\nabla g_j(x_0)]^t \theta_j(x, x_0) \geq 0 \Rightarrow \sum_{j=1}^k \lambda_j g_j(x_0) \leq 0, \forall x \in X. \quad (3.8)$$

Définition 3.5 We say the problem (PV) is of pseudo quasi V-type I at $x_0 \in X$ with respect to $(\eta_i)_{i=\overline{1,N}}$ and $(\theta_j)_{j=\overline{1,k}}$, if there exists $N + k$ vector functions $\eta_i : X \times X \rightarrow \mathbb{R}^n$, $i = \overline{1,N}$ and $\theta_j : X \times X \rightarrow \mathbb{R}^n$, $j = \overline{1,k}$ such that for some vectors $\mu \in \mathbb{R}_{\geq}^N$ and $\lambda \in \mathbb{R}_{\geq}^k$:

$$\sum_{i=1}^N \mu_i [\nabla f_i(x_0)]^t \eta_i(x, x_0) \geq 0 \Rightarrow \sum_{i=1}^N \mu_i [f_i(x) - f_i(x_0)] \geq 0, \forall x \in X, \quad (3.9)$$

$$\sum_{j=1}^k \lambda_j g_j(x_0) \geq 0 \Rightarrow \sum_{j=1}^k \lambda_j [\nabla g_j(x_0)]^t \theta_j(x, x_0) \leq 0, \forall x \in X. \quad (3.10)$$

3.2 Optimality conditions

In this section we establish some sufficient conditions for an $x_0 \in X$ to be an efficient or properly efficient solution of problem (PV) under different conditions while involving the defined notion of problem given above.

Théorème 3.1 (Sufficiency) Suppose that

- (i) $x_0 \in X$;
- (ii) there exists $N + J$ vector functions $\eta_i : X \times X \rightarrow \mathbb{R}^n$, $i = \overline{1,N}$, $\theta_j : X \times X \rightarrow \mathbb{R}^n$, $j \in J(x_0)$ and scalars $\mu_i > 0$, $i = \overline{1,N}$, $\lambda_j \geq 0$, $j \in J(x_0)$ such that :

$$\sum_{i=1}^N \mu_i [\nabla f_i(x_0)]^t \eta_i(x, x_0) + \sum_{j \in J(x_0)} \lambda_j [\nabla g_j(x_0)]^t \theta_j(x, x_0) \geq 0, \quad \forall x \in X; \quad (3.11)$$

(iii) the problem (PV) is V-type I at x_0 with respect to $(\eta_i)_{i=\overline{1, N}}$, $(\theta_j)_{j \in J(x_0)}$ and for μ and λ .

Then x_0 is properly efficient solution for (PV).

Théorème 3.2 (Sufficiency) Suppose that

(i) $x_0 \in X$;

(ii) there exists $N + J$ vector functions $\eta_i : X \times X \rightarrow \mathbb{R}^n$, $i = \overline{1, N}$, $\theta_j : X \times X \rightarrow \mathbb{R}^n$, $j \in J(x_0)$ and scalars $\mu_i \geq 0$, $i = \overline{1, N}$, $\sum_{i=1}^N \mu_i = 1$, $\lambda_j \geq 0$, $\forall j \in J(x_0)$ such that the relation (3.11) of theorem 3.1 is verified;

(iii) the problem (PV) is strictly-pseudo V-type I at x_0 with respect to $(\eta_i)_{i=\overline{1, N}}$, $(\theta_j)_{j \in J(x_0)}$ and for μ and λ .

Then x_0 is an efficient solution of (PV).

If further $\mu_i > 0$, $i = \overline{1, N}$, x_0 will be properly efficient solution for (PV).

Théorème 3.3 (Sufficiency) Suppose that

(i) $x_0 \in X$;

(ii) there exists $N + J$ vector functions $\eta_i : X \times X \rightarrow \mathbb{R}^n$, $i = \overline{1, N}$, $\theta_j : X \times X \rightarrow \mathbb{R}^n$, $j \in J(x_0)$ and scalars $\mu_i \geq 0$, $i = \overline{1, N}$, $\sum_{i=1}^N \mu_i = 1$, $\lambda_j \geq 0$, $\forall j \in J(x_0)$ such that the relation (3.11) of theorem 3.1 is verified;

(iii) the problem (PV) is quasi strictly-pseudo V-type I (or semi strictly-quasi V-type I) at x_0 with respect to $(\eta_i)_{i=\overline{1, N}}$, $(\theta_j)_{j \in J(x_0)}$ and for μ and λ .

Then x_0 is an efficient solution for (PV).

To prove the next result we need to define the generalized assumption C at a point $x^0 \in X$. Condition imposed on the functions η_i , $i = \overline{1, m}$.

Generalized Assumption C at a point $x^0 \in X$: Let $\eta_i : X \times X \rightarrow \mathbb{R}^n$, $i = \overline{1, m}$, $\eta : X \times X \rightarrow \mathbb{R}^n$ and $x^0 \in X$. We say that the functions η_i , $i = \overline{1, m}$ satisfy the generalized assumption C at x^0 with respect to η if, for any $x \in X$ and for any $\alpha \in [0, 1]$,

$$\eta_i(x^0, x^0 + \alpha\eta(x, x^0)) = -\alpha\eta(x, x^0), \quad i = \overline{1, m}, \quad (3.12)$$

$$\eta_i(x, x^0 + \alpha\eta(x, x^0)) = (1 - \alpha)\eta(x, x^0), \quad i = \overline{1, m}. \quad (3.13)$$

Théorème 3.4 (necessity) Suppose that

- (i) x^0 is a weak efficient solution for (PV);
- (ii) g_j is continuous at x^0 for $j \in \tilde{J}(x^0)$, there exists vector functions $\eta_i : X \times X \rightarrow \mathbb{R}^n$, $i = \overline{1, N}$, $\theta_j : X \times X \rightarrow \mathbb{R}^n$, $j \in J(x^0)$ continuous in the second argument at x^0 and which satisfy the inequality (3.13) of generalized assumption C at x^0 with respect to $\eta : X \times X \rightarrow \mathbb{R}^n$;
- (iii) the functions $[\nabla f_i(x^0)]^t \eta_i(x, x^0)$, $i = \overline{1, N}$ and $[\nabla g_j(x^0)]^t \theta_j(x, x^0)$, $j \in J(x^0)$ are pre-invex of x on X with respect to $\gamma : X \times X \rightarrow \mathbb{R}^n$;
- (iv) there exists $\bar{x} \in X$ such that

$$-g_j(x^0) > [\nabla g_j(x^0)]^t \theta_j(\bar{x}, x^0), \quad \forall j \in J(x^0). \quad (3.14)$$

Then there exists $\mu^0 \in \mathbb{R}_{\geq}^N$ and $\lambda^0 \in \mathbb{R}_{\geq}^{|J(x^0)|}$ such that (x^0, μ^0, λ^0) satisfies

$$\sum_{i=1}^N \mu_i^0 [\nabla f_i(x^0)]^t \eta_i(x, x^0) + \sum_{j \in J(x^0)} \lambda_j^0 [\nabla g_j(x^0)]^t \theta_j(x, x^0) \geq 0, \quad \forall x \in X. \quad (3.15)$$

3.3 Mond-Weir Duality

In relation to (PV) we consider the following multiobjective dual problem, which is in the forma of Mond-Weir.

$$(D) \quad \text{Maximize } f(y) = (f_1(y), \dots, f_N(y)),$$

subject to

$$\sum_{i=1}^N \mu_i [\nabla f_i(y)]^t \eta_i(x, y) + \sum_{j \in J(y)} \lambda_j [\nabla g_j(y)]^t \theta_j(x, y) \geq 0, \quad \forall x \in X, \quad (3.16)$$

$$\mu \in \mathbb{R}_{\geq}^N, \quad \lambda \in \mathbb{R}_{\geq}^k, \quad \eta_i : X \times X \rightarrow \mathbb{R}^n, \quad \forall i = 1, \dots, N, \quad \theta_j : X \times X \rightarrow \mathbb{R}^n, \quad \forall j \in J(y).$$

We let Y be the set of feasible solutions of problem (D).

Théorème 3.5 (Weak duality) Suppose that

- i) $x \in X$;
- ii) $(y, \mu, \lambda, (\eta_i)_{i=\overline{1, N}}, (\theta_j)_{j \in J(y)}) \in Y$;
- iii) the problem (PV) is pseudo-quasi V-type I at y with respect to $(\eta_i)_{i=\overline{1, N}}, (\theta_j)_{j \in J(y)}$ and for μ and λ .

Then $f(x) \not\leq f(y)$.

Théorème 3.6 (Weak duality) Suppose that

- i) $x \in X$;
- ii) $(y, \mu, \lambda, (\eta_i)_{i=\overline{1, N}}, (\theta_j)_{j \in J(y)}) \in Y$;
- iii) the problem (PV) is strictly pseudo-quasi V-type I (or quasi strictly pseudo V-type I) at y with respect to $(\eta_i)_{i=\overline{1, N}}, (\theta_j)_{j \in J(y)}$ and for μ and λ .

Then $f(x) \not\leq f(y)$.

Théorème 3.7 (Strong duality) Suppose that

- 1) x^0 is a weak efficient solution for (PV) ;
- 2) the hypothesis (ii), (iii) and (iv) of theorem 3.4 are satisfied.

Then there exists $\mu^0 \in \mathbb{R}_{\geq}^N$ and $\lambda^0 \in \mathbb{R}_{\geq}^{|J(x^0)|}$ such that $(x^0, \mu^0, \lambda^0, (\eta_i)_{i=\overline{1, N}}, (\theta_j)_{j \in J(x^0)}) \in Y$ and the objective functions of (PV) and (D) have the same values at x^0 and $(x^0, \mu^0, \lambda^0, (\eta_i)_{i=\overline{1, N}}, (\theta_j)_{j \in J(x^0)})$, respectively. If, further, the problem (PV) is pseudo-quasi V-type I at all feasible solutions of (D), then $(x^0, \mu^0, \lambda^0, (\eta_i)_{i=\overline{1, N}}, (\theta_j)_{j \in J(x^0)}) \in Y$ is a weak efficient solution of (D).

Théorème 3.8 (Strong duality) Suppose that

- 1) x^0 is a weak efficient solution for (PV) ;
- 2) the hypothesis (ii), (iii) and (iv) of theorem 3.4 are satisfied.

Then there exists $\mu^0 \in \mathbb{R}_{\geq}^N$ and $\lambda^0 \in \mathbb{R}_{\geq}^{|J(x^0)|}$ such that $(x^0, \mu^0, \lambda^0, (\eta_i)_{i=\overline{1, N}}, (\theta_j)_{j \in J(x^0)}) \in Y$ and the objective functions of (PV) and (D) have the same values at x^0 and $(x^0, \mu^0, \lambda^0, (\eta_i)_{i=\overline{1, N}}, (\theta_j)_{j \in J(x^0)})$, respectively. If, further, the problem (PV) is strictly pseudo-quasi V-type I (or quasi strictly pseudo V-type I) at all feasible solutions of (D), then $(x^0, \mu^0, \lambda^0, (\eta_i)_{i=\overline{1, N}}, (\theta_j)_{j \in J(x^0)}) \in Y$ is an efficient solution of (D).

Théorème 3.9 (Converse duality) Suppose that

- (i) $(y^0, \mu^0, \lambda^0, (\eta_i)_{i=\overline{1, N}}, (\theta_j)_{j \in J(y^0)}) \in Y$ with $\mu^0 > 0$;
- (ii) $y^0 \in X$;
- (iii) the problem (PV) is V-type I (or semi strictly pseudo V-type I for g or pseudo quasi V-type I) at y^0 with respect to $(\eta_i)_{i=\overline{1, N}}, (\theta_j)_{j \in J(y^0)}$ and for μ^0 and λ^0 .

Then y^0 is properly efficient solution for (PV).

Théorème 3.10 Suppose that

- i) $x^0 \in X$ and $(y^0, \mu^0, \lambda^0, (\eta_i)_{i=\overline{1, N}}, (\theta_j)_{j \in J(y^0)}) \in Y$ such that

$$\sum_{i=1}^N \mu_i^0 f_i(x^0) = \sum_{i=1}^N \mu_i^0 f_i(y^0); \quad (3.17)$$

- ii) the problem (PV) is strictly pseudo quasi V-type I at y^0 with respect to $(\eta_i)_{i=\overline{1, N}}, (\theta_j)_{j \in J(y^0)}$ and for μ^0 and λ^0 .

Then $x^0 = y^0$.

3.4 Conclusion

In this paper, we have defined new classes of problem called V-type I with respect to $(\eta_i)_i$ and $(\theta_j)_j$, quasi-, pseudo-, pseudo quasi-, quasi pseudo- V-type I with respect to $(\eta_i)_i$ and $(\theta_j)_j$, as a generalization of type I functions with respect to η [1, 5] and the V type I problem with respect to η [4]. We have obtained necessary and sufficient conditions for a feasible point to be a weak efficient, efficient or properly efficient solution, we have defined the dual problem and we have proved weak, strong and converse duality theorems, involving the above classes of problems. For establishing the necessary condition for a feasible point to be a weak efficient solution, we have defined the generalized assumption C for the functions $(\eta_i)_i$ and $(\theta_j)_j$. Known results in the literature [4, 5] can be deduced as particular cases from our results, when the functions $(\eta_i)_i$ and $(\theta_j)_j$ are equal to the same function η which is different to zero.

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