

Polynomial Chaos Expansion for Uncertainty Propagation In Unreliable Queueing Models

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Doctoriales de Recherche Opérationnelle, le 12 et 13 Décembre 2018



I. Introduction

Uncertainty exists in almost all mathematical models we use to simulate the real world. Uncertainty is typically separated into aleatory uncertainty and epistemic uncertainty [1]. The aleatory uncertainty is due to the presence of some inherent non-deterministic characteristics of the process under analysis and it is also often called stochastic uncertainty. The epistemic uncertainty, often called subjective uncertainty is an uncertainty representing our lack of knowledge of (of some) the characteristics of the system under analysis. The role of uncertainty analysis is the study of the propagation of uncertainty from the input to the output through the system shown in Fig. 1.

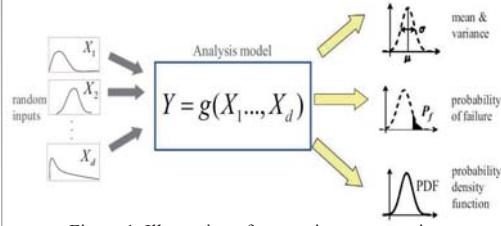


Figure 1. Illustration of uncertainty propagation

II. Methodology

Recent developments in uncertainty propagation (UP) have explored the use of polynomial chaos (PC), based on a suitable choice of orthogonal polynomial and its associated stochastic process, for stochastic representation of uncertainty

As a well-known and rigorous probabilistic UP approach, the polynomial chaos expansion (PCE) technique has been widely studied and applied recently in several fields, due to its ability to generate an analytical function for the uncertain random variables.

II. 1. Modeling uncertainty in queueing system via PCE

The objective of the method is to assess how uncertainties of inputs of queueing systems manifest in its outputs. To see how it works, consider the M/G/1/N queue with server subject to breakdowns and repairs where only one input parameter is uncertain.

Basic assumptions :

The basic assumptions of the model are described as follows :

- (1) Customers arrive at the system according to a Poisson process with parameter λ .
- (2) Service times are general and independent identically distributed with mean $1/\mu$.
- (3) Customers attempting to enter the queue when there are already N customers present are lost.
- (4) The server can break down with probability θ .

- (5) Repair times are exponentially distributed with rate r .
- (6) The server can break down only at the beginning of a service.

This system is modeled by a Markov chain X_n embedded at service completions and completion of a repair and has state space $S = \{0, 1, \dots, N-1\}$. Note that the Markov chain X_n is unichain and therefore the stationary distribution exists. Let π denote the stationary distribution of the M/G/1/N queue with breakdowns and repairs. Assume that the probability θ is the uncertain parameter of model, θ can be expressed in terms of central value $\hat{\theta}$, which when not accounting for uncertainty would be used as input, and a spread σ characterizing the likely range of values around θ :

$$\theta(\omega) = \hat{\theta} + \sigma \varepsilon(\omega) \quad (1)$$

where $\varepsilon(\omega)$ is considered as random noise inflicted on θ , it modelizes epistemic uncertainty. The propagation of the uncertainty in the input θ through the functional $\pi(\theta(\omega))$, leads to a random response of the system. This random response denoted by $Y = \pi(\theta)$.

II. 2. Polynomial chaos expansion

The method centers on the assumption that the random response Y can be efficiently described by series of polynomials of the input $\theta(\omega)$. Provided Y has a finite variance, and $\pi(\theta) \in L_2(\Omega, \mathcal{A}, P)$, it can be expressed in an orthogonal polynomial basis as follows :

$$\pi_n(\theta) \approx \sum_{i=0}^{P_p} y_i \Psi_i(\theta), \quad (2)$$

where $\{\Psi_i(\theta)\}_{0 \leq i \leq P_p}$ form an orthogonal polynomials basis, and y_i are the coefficients of the approximating series expansion, to be determined. The polynomials are chosen based on the distribution of the random variable θ . For example, for the normal distribution, the corresponding orthogonal polynomial is the Hermite polynomial.

II. 3. Polynomial chaos coefficients

The way to compute the PC coefficients $\{y_i\}$, can be casted into two different families : projection methods and regression methods [2]. Here we have used the first one which exploits the orthogonality of the PCE basis.

By premultiplying Eq. (2) by $\Psi_i(\theta)$ and by taking the expectation of the product, one gets the exact expression of each coefficient y_i :

$$y_i = E[\pi_n(\theta) \Psi_i(\theta)] = \int_{D \in R} \pi_n(\theta) \Psi_i(\theta) p_\theta(\theta) d\theta \quad (3)$$

Once the PCE in Eq.(2) is fully defined the statistical moments of the random processes can be easily calculated.

The first two moments of Y are obtained using the properties of the orthogonal polynomials :

$$\begin{cases} E[Y] = y_0 \\ Var[Y] = \sigma^2[Y] = E[Y^2] - E^2[Y] = \sum_{i=1}^{P_p} y_i^2 \\ Skew[Y] = \frac{E[Y^3] - 3E[Y]\sigma^2[Y] - E^3[Y]}{\sigma^3[Y]} \\ Kurt[Y] = \frac{E[Y^4] - 4E[Y]E[Y^3] + 6E^2[Y]\sigma^2[Y] + 3E^4[Y]}{\sigma^4[Y]} \end{cases} \quad (4)$$

III. Numerical results

For numerical computations, we propose one type of distribution for the service time, namely Erlang (E_2) of second order. The parameters N , λ and r are deterministic and their values are set : $N = 7$, $\lambda = 1$ and $r = 0.4$, respectively. θ is given by :

$$\theta(\omega) = 0.5 + 0.28 \varepsilon(\omega), \quad \varepsilon(\omega) \sim U[-1, 1]$$

We determine the stationary distribution vector with respect to the random variable $\theta(\omega)$. Therefore, the random outputs of interest are the stationary distributions $\pi_i(\theta(\omega))$, $i \in \{0, \dots, 6\}$. Their projection is performed on a univariate PC basis of degree $n = 4$. Numerical results for the statistical moments using Monte-carlo simulation of sample size $N=100\ 000$ are given in Tab. 1. Results are compared with those of the PCE.

Table 1. Statistical moments of steady-state vector in M/ E_2 /1/7

$\pi(\theta)$	$E[\pi_n(\theta)]$		$V[\pi_n(\theta)]$		$Skew(\pi_n(\theta))$		$Kurt(\pi_n(\theta))$	
	PCE	MC	PCE	MC	PCE	MC	PCE	MC
π_0	0.0237	0.0230	1.7753	1.7809	0.5053	0.5064	2.0384	2.0341
π_1	0.0413	0.0422	3.6413	3.6423	0.3565	0.3550	1.8968	1.8953
π_2	0.0660	0.0667	5.2349	5.2175	0.1730	0.1701	1.7864	1.7900
π_3	0.1018	0.1026	5.0800	5.0508	-0.0722	-0.0730	1.7463	1.7495
π_4	0.1557	0.1563	1.8737	1.8732	-0.5848	-0.5922	2.0301	2.0385
π_5	0.2393	0.2401	2.1440	2.1534	-0.7140	-0.7036	2.1921	2.1972
π_6	0.3722	0.3719	58.769	58.719	0.0543	0.0540	1.7759	1.7724

IV. REFERENCES

- [1]. Aleti, A., Trubiani, C., van Hoorn, A. and Jamshidi, P., 2018. An efficient method for uncertainty propagation in robust software performance estimation. *Journal of Systems and Software*, 138, pp.222-235.
- [2]. Wu, W., Bulian, G., McCue, L.S., Wu, W., Bulian, G. and McCue, L., 2011. Uncertainty analysis for parametric roll using non-intrusive polynomial chaos. In *Proceedings of the 12th international ship stability workshop*, Washington, DC, USA.