

## Diversification in Portfolio Selection Problems: A Definition and some Financial Consequences.

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**Abstract:** The purpose of this article is to propose a general definition of « more diversified » in portfolio problems and relate this definition to the optimal investment strategy. Central to this concern is the identification of what it is that is diversified and what determine the desirable degree of diversification. Some partial diversification results are provided using both the general expected utility model for arbitrary distributions as well as standard parametric distributions suitable for portfolio theory emphasizing the role of different parameters in determining the degree of diversification. It is shown that if the random vector of returns  $Y$  is a mutual fund transformation of  $X$  then every risk averse investor diversify more under  $X$ . Furthermore, for elliptical and skew-normal distributions, necessary and sufficient conditions are given to insure that short sales are not the optimal investment strategy for all risk-averse investors. We also discuss the relation between relevant parameters of two distributions that imply more diversification under one distribution rather than the other.

**Key words:** Portfolio selection, Diversification, Majorization, Negative interdependence, Elliptical distributions, Skew-elliptical distributions, Separating distributions.

### 1/ Introduction

An important motivation for the theory of portfolio choice is the phenomenon of diversification. When faced with the problem of allocating wealth to  $n$  alternative (but not mutually exclusive) risky investments, most investors do not choose a unique asset but they diversify their holdings among the  $n$  assets. The theory has developed from a need to rationalize

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diversification as an optimal investment strategy and from a need to derive the financial consequences of this strategy. As explained by Scheffman (1975), the interest in knowing whether investors would tend to diversify (hold all assets long), to concentrate (hold only one asset) or take short positions has important implications for the "size" and "depth" of the capital markets and the effects of various changes on these markets.

The mean variance theory of portfolio selection of Morkowitz (1952) and Tobin (1958) and its market equilibrium extension, the capital asset pricing model (CAPM), of Sharpe (1964), Lintner (1965) and Mossin (1966) provided an explanation of the diversification phenomenon.

Investors, according to the mean variance theory, dislike variance and seek to reduce it by diversifying their holdings among the alternative assets given their subjective return to risk tradeoff. At equilibrium, investors are compensated by a higher mean only for bearing systematic risk; all nonsystematic risk is diversified away. In a world with a riskless asset, the extent of diversification in a particular portfolio of risky assets is measured by the correlation coefficient with the market portfolio, which is the most diversified by definition. The theory does not provide an explicit definition of diversification.

An important implication of the mean-variance analysis, due to Morkowitz (1952) and Tobin (1958), is the two fund separation theorem. Given a world with a riskless asset, the optimal portfolio of every risk averse investor is simply a subjective combination of the same fund of risky assets and the riskless asset. In a world without a riskless asset, Black (1972) generalized the two fund separation theorem using two efficient funds of risky assets.

Criticisms of the mean variance analysis stems from the objections to variance as a measure of risk (see Rothschild & Stiglitz (1970), Hadar & Russel (1969) and Hanoch & Levy (1969)), to the use of quadratic utilities (see Arrow (1971) and Samuelson (1967b)), to the assumption of normality of the distribution of assets returns (see Fama (1965b), Samuelson (1967a), Blattberg & Gonedes (1974), Longin (1995), Duffie & Pan (1997)) and to the correlation coefficient as a model of dependence between asset returns (see Samuelson (1967b)), Brumelle (1974), Scheffman (1975) and Epstein & Tanny (1980)).

To demonstrate the validity of the mean-variance analysis, Samuelson (1970) has shown that it is a good approximation for "compact" distributions and Merton (1971) has justified it in intertemporal portfolio problems when trading takes place continuously. Chamberlain (1983) used the elliptical class of distributions to characterize asset returns distributions

that imply mean-variance utilities. Owen and Rabinovitch (1983) showed that elliptical distributions extend the Tobin (1958) separation theorem and the results of CAPM to non normal symmetric distributions which are not necessarily stable.

Touati-Tliba (2003) discussed, within the mean-variance framework, differences between normal and non normal elliptical distributions, especially those with heavier tails than normal, focusing on the role of Kurtosis, as a measure of fatness of tails, in determining the optimal investment strategy of risk-averse investors

Extensions of the basic mean-variance theory to include higher order moments can be found in Rubinstein M. (1973), Kraus and Litzenberger (1976), and more recently in Jurczenko & Maillet (2001), Simaan (1993a), Adcock & Shutes (1999) and Adcock (2002).

Rubinstein M. (1973) derived an equation for expected return in terms of an arbitrary number of co-moments. Unlike Kraus and Litzenberger (1976) and Jurczenko & Maillet (2001) who used a utility justification for a mean-variance-skewness framework, Simaan (1993a), Adcock & Shutes (1999) and Adcock (2002) used more general laws.

Simaan (1993a) developed a three parameter normative portfolio analysis where idiosyncratic security risks are modeled as following a joint elliptical distribution and skewness is generated by a single factor for the whole economy upon which different securities have different loadings. He showed that his generating process results in a three fund separation and provides a three parameter CAPM.

Adcock & Shutes (1999) used the multivariate skew-normal distribution, developed first in the statistics literature by Azzalini & Dalla Valle (1996), to develop a mean-variance-skewness framework. In order to incorporate both kurtosis and skewness, Adcock (2002) presents an analysis based on the multivariate skew-student distribution.

Ross (1978) described the necessary and sufficient conditions on distributions of asset returns for portfolio choice to be equivalent to choice among a limited number of mutual funds, independent of investors' preferences. He called such distributions the separating distributions and asserted that his work is the dual of the work of authors like Cass & Stiglitz (1970) who have developed conditions on utility functions which imply separation for all distributions.

The issue of diversification is central to the analysis of portfolio selection problems. Most of the early diversification results in portfolio

analysis can be found in Samuelson (1967b). Within the context of the general expected utility model, many authors' derived sufficient conditions on the distributions of asset returns under which diversification is the optimal investment strategy for all risk adverse investors.

Samuelson (1967b), Davies and Ronning (1974) and Hadar & Russel (1971, 1974) showed that exchangeability (symmetry of the joint distribution of asset returns) is sufficient for total diversification in the sense that the proportions invested in each asset are equal for all risk averse investors.

In other studies, mostly in a two assets world, various notions of negative interdependence in asset returns were presented and shown, when the two assets have equal means, to be sufficient for positive diversification i.e. holding assets long. In fact, as argued by Brumelle (1974), negative correlation is neither necessary nor sufficient for diversification, except in special cases implying mean-variance framework.

Samuelson (1967b) provided a sufficient condition for positive diversification using the concept of conditional probability.

Another notion of negative interdependence was introduced by Brumelle (1974) and shown to be sufficient for diversification in the two assets case.

Scheffman (1975) defined a new concept of generalized correlation (G-correlation) between asset returns and showed that all risk-averse investors hold positive amounts in each asset in the case of negative G-correlation.

Epstein and Tanny (1980) took the analysis further by turning from the question of the optimality of diversification to an examination of the degree of diversification. They characterized « more negatively interdependent » between two random variables (assets) through the notion of "elementary correlation increasing transformation" (CIT) and asked if CIT induces a risk-averse investor to diversify more « in the sense that he will divide his total investment more equally between the two assets. » (See Epstein and Tanny 1980, p. 125)

Their definition allows an answer to this question, in the case of equal means, only in special cases, for example, when utilities are quadratic or when the bivariate distribution is normal.

It should be noted that, for diversification, the equality of the means of asset returns is essential. Of course, exchangeability implies equal means. On the other hand, for positive diversification, Brumelle (1974) and Hadar & Russel (1971), for example, have noted that if all risk averse investor should diversify then the means are necessarily equal.

In an empirical study, Levy (1979) found that diversification between industries is preferable to a random diversification given a short-term horizon, e.g. one year or less.

Conine & Tamarkin (1981) state that total diversification is the rational investment strategy for a risk averse investor in a homogeneous market if he is concerned only with the first two moments of asset returns. To rationalize the observed behavior of market participants which shows that the majority of individual investors hold imperfectly diversified portfolios, they add that "Anti diversification may obtain because increasing the number of risk assets in a portfolio not only will reduce variance, a positive consideration, but also will reduce portfolio skewness, a negative consideration".

The purpose of this research is to reexamine the issue of diversification by proposing a general definition of « more diversified » in portfolio problems and relating this definition to the optimal investment strategy. Central to this concern is the question of what it is that is diversified and what the desirable degree of diversification is. Some partial diversification results are provided using both the general expected utility model for arbitrary distributions as well as standard parametric distributions which proved to be useful in portfolio theory emphasizing the role of different parameters (mean, variance, skewness,...) in determining the degree of diversification in the portfolios of risk averse investors.

## 2/ Diversification in portfolio selection problems: A definition

Consider an investor with initial wealth  $w_0 > 0$  to be invested in  $n$  risky assets available in the market and indexed by the set  $N = \{1, 2, \dots, n\}$ ,  $\forall i \in N$  define  $X_i$  to be one plus the uncertain rate of return on the asset  $i$  and  $a_i$  is the proportion of wealth invested in asset  $i$  the portfolio problem is:

$$\max_{a \in \Lambda} E(u(w_0 a' X))$$

Where  $\Lambda = \{\lambda \in R^n / \lambda' e = 1\}$  denote the set of all feasible portfolios,  $e \in R^n$  is a vector of ones and prime denotes the transpose of a vector.

We assume that assets are infinitely divisible and investors are risk-averse expected utility maximizers. The function  $u(\cdot)$  is assumed to be increasing, concave and at least twice continuously differentiable, when  $u$  satisfies these conditions we write  $u \in U$ .

The assumption of risk aversion is standard in the economics of uncertainty literature, it is more so in portfolio analysis. If we drop this requirement and assume risk neutrality instead then the portfolio problem as defined above has the form of a continuous knapsack problem. Hence, when short sales

are allowed then the problem is unbounded, otherwise the optimal strategy is the concentration of investment in the asset that has the highest mean. Note that short sales are allowed here because among other reasons<sup>2</sup> we want to examine the extent of diversification, imposing restrictions on short sales would imply imposing a degree of diversification.

Before discussing diversification in the context of portfolio problem we need to introduce first some notations and establish some definitions.

For a scalar  $r \in R$ , define the simplex  $\Lambda_r = \{\lambda \in R^n / \lambda'e = r\}$ , on  $\Lambda_r$  we define the preorder relation  $\prec_m$  as follows:

**Definition 1:** We say that  $a$  is majorized<sup>3</sup> by  $b$  and write  $a \prec_m b$  if and only if:

(i)  $a \in \Lambda_r$  and  $b \in \Lambda_r$  for some  $r \in R$

(ii)  $\sum_{i=1}^k a_i^* \leq \sum_{i=1}^k b_i^* \quad k \in N = \{1, 2, \dots, n\}$

where  $a^*$  and  $b^*$  are the vectors  $a$  and  $b$  rearranged in a decreasing order.

**Remarks:**

1/ the relation  $\prec_m$  is only a preorder. It is not anti symmetric, in fact  $a \prec_m b$  and  $b \prec_m a$  imply  $a = b\Pi$  for some permutation matrix  $\Pi$ .

2/ it can be shown<sup>4</sup> that the following statements are equivalent:

i)  $a \prec_m b$

ii)  $a = Qb$  where  $Q$  is some doubly stochastic<sup>5</sup> matrix

iii)  $a \in \text{convex hull}\{b\Pi_i, i = 1, 2, \dots, n!\}$  where  $\Pi_i$  are the permutation matrices.

iii)  $a$  can be derived from  $b$  by successive applications of a finite number of  $T$  transformations; a  $T$  transformation is given by  $T_{ij} = \alpha I + (1 - \alpha)Q_{ij}$   $\alpha \in [0, 1]$  and  $Q_{ij}$  is a permutation matrix that interchanges the  $i^{\text{th}}$  and the  $j^{\text{th}}$  coordinates and  $I$  is the identity matrix.

<sup>2</sup> With restriction on short sales, it is quit difficult to characterize the mean-variance efficient set of portfolios. Dybvig (1985) showed that kinks in the efficient set are the rule rather than the exception if short sales are not allowed.

<sup>3</sup> See A. Marshall & I. Olkin (1979)

<sup>4</sup> See Marshall & I. Olkin (1979)

<sup>5</sup> The matrix  $Q$  is doubly stochastic if and only if  $q_{ij} \geq 0$ ,  $e'Q = e'$  and  $Qe = e$ .

3/ If the function  $\psi: R^n \rightarrow R$  is increasing with respect to  $\prec_m$ , it is said to be Schur-convex and if  $a \prec_m b \Rightarrow \psi(a) > \psi(b)$  then  $\psi$  is said to be Schur-concave.

Epstein and Tanny (6) raised an important question about the nature of interdependence between assets that induces a risk-averse investor to diversify more in the sense that he will divide his total investment more equally between these assets. This question raises the need to establish a general definition of "greater diversification" in the context of portfolio selection and then to seek the conditions that imply more diversification.

We propose the following definition<sup>6</sup>:

**Definition 2:** Let  $a, b \in \Lambda$ ,  $a$  is said to be "more diversified than"  $b$  if and only if  $a \prec_m b$ .

This definition is, on one hand, very intuitive. Portfolio  $a$  is more diversified than portfolio  $b$  means that the sum of the  $k$  largest proportions of portfolio  $a$  is not larger than that of portfolio  $b$  for all  $k \in N = \{1, 2, \dots, n\}$ . On the other hand, it is appealing because it is independent of investors' preferences as well as of returns distributions.

Furthermore, it is potentially useful in providing a unified conceptual framework to the theory of diversification in portfolio selection problems as it provided a unified framework to the theory of inequalities in mathematics. Of particular interest, in the portfolio problem context, are the following three sets:

$\Lambda = \{\lambda \in R^n / \lambda'e = 1\}$  the set of all feasible portfolios with no restriction

$\Lambda_+ = \{\lambda \in R_+^n / \lambda'e = 1\}$  the set of all portfolios with restriction on short sales:

$\Lambda_0 = \{\lambda \in R^n / \lambda'e = 0\}$  the set of arbitrage portfolios:

Note that: 1/ If  $a \in \Lambda$  then  $\frac{1}{n}e \prec_m a$

2/ If  $a \in \Lambda_+$  then  $\frac{1}{n}e \prec_m a \prec_m e^i$  where  $e_j^i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

3/ If  $a \in \Lambda_0$  then  $0 \prec_m a$

<sup>6</sup> I am indebted to professor D. Nachman for this definition.

Clearly, the portfolio  $\frac{1}{n}e$  is the most diversified and the portfolio  $e^i, i \in N$ , is the most concentrated if short sales are not allowed. Note that all the usual diversification forms, common in the literature, might be expressed in terms of majorization. We may express the requirement of total diversification by writing  $a \prec_m \frac{1}{n}e$  which imply necessarily  $a = \frac{1}{n}e$ . Similarly, for non negative diversification, when short sales are not desirable or allowed, we write  $a \prec_m e^i$ . Likewise if we want to exclude concentration and impose some degree of positive diversification we write  $a \prec_m e^i$  and  $a \neq e^i$ . Furthermore, this definition allows expressing naturally the question of greater diversification between portfolios. In addition, this notion of "more diversification" is related to the well known Lorenz curve as applied to the distribution of wealth between alternative assets in a particular portfolio.

Define  $s_k = \sum_{i=1}^k a_i^*$   $k \in N$ . The points  $\left\{ \left( \frac{k}{n}, s_k \right) / k \in N \right\}$  define the Lorenz curve.

Given the definition above, we reexamine the issue of diversification. In particular, we want to characterize the relation between asset returns distributions that make all risk adverse investors diversify more under one distribution than under the other.

### 3/ Some diversification results in portfolio selection problems

For the random vector of returns  $X \in R^n$  and the doubly stochastic non singular matrix  $Q$ , we define the transformation  $Y = QX$ . To interpret this, notice that  $Y$  is a vector of portfolios of  $X$  we may then say that  $Y$  is a mutual fund transformation<sup>7</sup> of  $X$ .

#### **Theorem 1: The mutual fund theorem.**

*Let  $X \in R^n$  be a random variable and let  $Y = QX$  where  $Q$  is a doubly stochastic matrix, we define the two problems:*

<sup>7</sup> The mutual fund transformation expresses the notion of stochastic majorization between two random vectors (see Marshall & Olkin).

$$P_X: \max_{a \in \Lambda} E(u(a'X)) \text{ and } P_Y: \max_{a \in \Lambda} E(u(a'Y)) \text{ where } u \in U.$$

If  $P_X$  has an optimal solution  $a_X$  then  $P_Y$  has an optimal solution  $a_Y$  that satisfies  $a_X \prec_m a_Y$ .

Proof of the mutual fund theorem:

Note that if  $Y = QX$  then  $\forall a \in \Lambda$  there exists a solution  $b \in \Lambda$  such that  $E(u(a'X)) = E(u(b'Y))$  and vice versa.

$$\text{Thus, } \max_{a \in \Lambda} E(u(a'X)) = \max_{a \in \Lambda} E(u(a'Y))$$

$$\text{Since } a_X \text{ is optimal for } X, \text{ we have: } E(u(a'_X X)) = \max_{a \in \Lambda} E(u(a'Y))$$

$$\text{Hence } E(u(a'_X Q^{-1}Y)) = \max_{a \in \Lambda} E(u(a'Y))$$

Thus the optimal portfolio for  $Y$  can be given by  $a_Y = Q'^{-1}a_X$

This implies  $a_X = Q'a_Y$  and hence  $a_X \prec_m a_Y$  End of proof

Note that the mutual fund transformation insures that the difference between  $X$  and  $Y$  that causes every risk averse investor to diversify more under  $X$  is not a result of a change in the overall return-risk choices but rather a result of a redistribution of these choices.

Whatever diversification means, the claim that mutual funds provide it should be valid. Hence the optimal portfolio under  $Y$  need not be as diversified as under  $X$ .

**Remark:** The notion that diversity implies more diversification is intuitively acceptable. Notice that  $X$  presents more diversity between assets than does  $Y$  and hence should imply more diversification.

In diversification studies, many researchers were concerned with developing models of dependence between random variables (asset returns). Positive (non negative) diversification has been associated with various notions of negative interdependence between assets.

Brumelle's (1974) notion of negative interdependence and Scheffman's (1975) concept of negative G-correlation as well as the correlation increasing transformation of Epstein and Tanny (1980) are all weaker sufficient conditions than the proposed definition of Samuelson's (1967b). It should be noted that all these notions imply that the correlation between the

two random variables, is negative. (See Scheffman p.278, Epstein and Tanny (1980) p.23 and the discussion in Brumelle p.481).

Brumelle argued that "the most appropriate definition of negative interdependence" in portfolio context "would be:  $x$  and  $y$  are negatively interdependent if diversification by all risk-averse investors is optimal." (See Brumelle (1974), p. 474). However his argument is unfortunate for it suggests that two (or more) positively correlated random variables are negatively associated even when dealing with a multivariate normal distribution where the correlation is unquestionably the appropriate model of dependence. For example, suppose  $\{X_i, i=1,2,\dots,n\}$  a set of random variables (asset returns) following a multivariate normal distribution with  $E(X_i)=\mu \forall i$ ,  $V(X_i)=\sigma^2 \forall i$  and  $\text{cor}(X_i, X_j)=\rho \forall i \neq j$  and let  $0 < \rho < 1$ , in fact we might take  $\rho=0.99$ , yet the unique optimal strategy to the portfolio problem, for all risk averse investors, is total diversification. It would be really unfortunate to claim that the random variables in this example are negatively interdependent. Of course, this is an example of exchangeability. Our argument, here, emphasizes the need to free the definitions of "diversification" and "greater diversification" from dependence on investors' preferences as well as from returns distributions characteristics.

In general, negative correlation is neither necessary nor sufficient for diversification. However, for special cases, for example when returns follow an elliptical distribution the usual correlation coefficient is all that is needed, as a model of dependence, to study diversification.

The class of elliptical distributions is examined in detail by Kelker (1970); it provides attractive appealing alternatives to normal distributions and includes stable and non-stable members. Detailed discussions of the relevance of the elliptical class to portfolio problems can be found in Owen and Rabinovitch (1983) and Chamberlain (1983).

Assuming the existence of the density,  $X \in R^n$  is said to be elliptically distributed, written  $X \sim \xi(\Delta, \Omega)$ , with location vector  $\Delta$  and characteristic matrix  $\Omega$  if its density is of the form:

$$f(X) = c_n |\Omega|^{-1/2} \phi((X - \Delta)' \Omega^{-1} (X - \Delta))$$

The function  $\phi(\cdot)$  is a positive function of a scalar variable and  $c_n$  is a constant of integration.

We assume throughout that  $X \sim \xi(\Delta, \Omega)$  possesses  $k > 2$  moments, hence  $E(X) = \Delta$  and the covariance matrix  $\Sigma = \gamma \Omega$  for some  $\gamma \in R_+$ . Furthermore,

assets are assumed to be linearly independent<sup>8</sup>, hence the matrix  $\Sigma$  is non-singular.

Under these conditions, Touati-Tliba (1983, 2003) established<sup>9</sup> that for all risk averse utilities,  $\forall u \in U$ , the optimal portfolio must satisfy:

$$a = \frac{\Sigma^{-1}e}{e'\Sigma^{-1}e} + \frac{e'\Sigma^{-1}\Delta}{M(w_0, a'X)} \left( \frac{\Sigma^{-1}\Delta}{e'\Sigma^{-1}\Delta} - \frac{\Sigma^{-1}e}{e'\Sigma^{-1}e} \right)$$

Where  $M(w_0, y) = \frac{-w_0 E(u''(w_0 y) l(w_0 y))}{E(u'(w_0 y))}$  with  $l(y) = \frac{h(y)}{f_y(y)}$ ,

$h(y) = \int_{-\infty}^y \frac{\Delta_y - x}{\sigma_y} f_y(x) dx$  and  $\Delta_y, \sigma_y^2$  and  $f_y(y)$  denote, respectively, the mean, the variance and the density function of  $Y$ .

Let  $a^m = \frac{\Sigma^{-1}e}{e'\Sigma^{-1}e}$ ,  $a^t = \frac{\Sigma^{-1}\Delta}{e'\Sigma^{-1}\Delta}$  and  $\eta = e'\Sigma^{-1}\Delta(a^t - a^m)$ . We see that the

portfolio  $a^m$  is the global minimum variance portfolio,  $a^t$  is a mean return to risk tradeoff portfolio and  $\eta$  is an arbitrage portfolio that uses no wealth since  $\eta'e = 0$ . Notice that the efficient set of portfolios is spanned by the two frontier portfolios<sup>10</sup>  $a^m$  and  $a^t$ . Thus the elliptical class implies two funds separation. However, if  $\Delta = ke$  where  $k \in R$ , this class implies one fund separation.

As a special case, when  $X \sim N(\Delta, \Sigma)$  and assuming constant absolute risk aversion (CARA) utility functions with  $\frac{-u''(y)}{u'(y)} = \alpha$ ,  $\alpha \in R_+$ , the optimal

solution<sup>11</sup> is given by:  $a(\alpha) = a^m + \frac{1}{\alpha w_0} \eta$  where  $w_0$  is the initial wealth.

The next theorem generalizes a known result for normal distributions<sup>12</sup> to the class of elliptical distributions.

<sup>8</sup> Linear independence means that  $P(\sum c_i x_i = 0) = 1 \Rightarrow c_i = 0 \forall i$ .

<sup>9</sup> See theorem 1 in Touati-Tliba (2003).

<sup>10</sup>  $a^t$  is a frontier portfolio, it is efficient only if  $e'\Sigma^{-1}\Delta > 0$

<sup>11</sup> These are the only conditions that imply a closed form solution. See Epps (1981)

<sup>12</sup> See Scheffman (1975), footnote 2 in p. 278.

**Theorem 2:** Let  $X \in R^n$  such that  $X \sim \xi(\Delta, \Omega)$  where the covariance matrix  $\Sigma = \gamma\Omega$  is non-singular, then the necessary and sufficient conditions for  $a^u \prec_m e^i$  (non negative (positive) diversification),  $\forall u \in U$  are:

i)  $\Delta = ke$  for some  $k \in R$

ii)  $\Sigma^{-1}e \geq (>)0$

Proof of theorem 2:

Without loss of generality set  $w_0 = 1$ . According to theorem 1, in Touati-Tliba (2003),  $\forall u \in U$  the optimal solution satisfies:

$$a^u = a^m + \frac{1}{M_u(a'X)} \eta \quad (1)$$

Where  $\eta = e'\Sigma^{-1}\Delta(a^t - a^m)$  and  $M_u(y) = \frac{-E(u''(y)y(y))}{E(u'(y))}$ .

In particular, for constant absolute risk aversion<sup>13</sup> (CARA) utility functions with  $\frac{-u''(y)}{u'(y)} = \alpha$ ,  $\alpha \in R_+$ , we have  $M_u(y) = M_\alpha(y) = \frac{-\alpha E(e^{-\alpha y}y(y))}{E(e^{-\alpha y})}$ .

1/ If  $\Delta = ke$  then  $a^m = a^t$  and  $\eta = 0$ . Thus we have  $a^u = a^m \quad \forall u \in U$ . But  $e'\Sigma^{-1}e > 0$  since  $\Sigma^{-1}$  is positive definite, hence  $\Sigma^{-1}e \geq (>)0$  implies  $a^u \geq (>)0 \quad \forall u \in U$

2/ If  $a^u \geq 0 \quad \forall u \in U$ , then  $a^\alpha = a^m + \frac{1}{M_\alpha(a'X)} \eta \geq 0 \quad \forall \alpha \in R_+$ , in scalar notation we have  $a_j^\alpha = a_j^m + \frac{1}{M_\alpha(a'X)} \eta_j \geq 0 \quad \forall \alpha \in R_+$

We show in this case that  $\eta = 0$ . For suppose  $\eta \neq 0$ , then there must exist  $j \in N$  such that  $\eta_j < 0$ , note that  $\lim_{\alpha \rightarrow 0} M_\alpha(y) = 0$  implies  $a_j < 0$  contrary to our assumption, hence  $\eta = 0$ .

But  $\eta = 0$  implies  $a^t = a^m$  hence  $\Delta = \frac{e'\Sigma^{-1}\Delta}{e'\Sigma^{-1}e} e = ke$

Furthermore, if  $\Delta = ke$  then  $a^u = a^m \quad \forall u \in U$  but this is nonnegative by assumption.

<sup>13</sup> If, for the particular elliptical distribution considered, the moment generating function did not exist, another parametric risk averse utility could be used with a choice of sufficiently small degree of risk aversion without altering the main argument in the proof.

Since  $e'\Sigma^{-1}e > 0$  we must have  $\Sigma^{-1}e \geq 0$       End of proof

For elliptical distributions, according to the theorem, short sales are not the optimal investment strategy for all risk-averse investors provided that means of asset returns are equal and  $\Sigma^{-1}$  have non negative row sums.

To introduce skewness and following Simaan (1993a), denote by  $X \in R^n$  the random vector of security returns, let  $\Delta$  and  $b$  be two arbitrary vectors and consider the following model of asset returns:

$$X = \Delta + by + Z$$

where conditional on  $y$  the random vector  $Z$  follows a joint elliptical distribution  $Z/y \sim \xi(0, \Omega)$  and  $y$  is a univariate random variable. When assets are uncorrelated, this model is the single factor (market) model. As argued by Simaan, given the flexibility of the elliptical class of distributions and the arbitrary choice of the distribution of  $y$ , this model allows individual assets to exhibit skewness and does not admit, in general, two-fund separation.

Note that if  $b = 0$  we retrieve the original elliptical distribution. On the other hand, if  $y$  follows an elliptical distribution then the joint distribution of  $X$  is itself elliptical (see Keller (1970)) which imply, according to Chamberlain (1983), a mean-variance expected utility. It is essential, therefore, to assume that  $b \neq 0$  and to specify a skewed distribution for  $y$  in order for individual assets to exhibit skewness and permit more than two-fund separation.

In the statistics literature, authors like Azzalini & Dalla Valle (1996) and Sahu et al. (2003), using transformation and conditioning, developed the skew-elliptical distributions.

By transforming  $(n+1)$  random variables following a multivariate normal distribution where one random variable is assumed to be independent of others then using conditioning on the independent variable being positive, Azzalini & Dalla Valle (1996) obtained the multivariate skew-normal distribution. Sahu et al. (2003) considered  $2n$  random variables following a multivariate elliptical distribution where the first  $n$  variables were assumed to be uncorrelated among themselves and with the others. They used transformation and then conditioning on the uncorrelated variables being all positive to derive another version of skew-elliptical distributions (including skew-normal distribution). The construction of these distributions with the conditioning introduces skewness.

In the finance literature, Adcock & Shutes (1999) used the multivariate skew-normal distribution to develop a mean-variance-skewness framework. In order to incorporate both kurtosis and skewness, Adcock (2002) presents an analysis based on the multivariate skew-student distribution. It should be noted that the one factor skew-elliptical distributions are special cases of the generating process specified by Simaan (1993a)

With some changes in the formulation of Azzalini & Dalla Valle (1996) and minor changes in that of Adcock & Shutes (1999), we reconsider the previous model

$$X = \Delta + by + Z$$

such that  $Z \sim N(0, \Sigma)$  with  $\Sigma$  non singular and  $y$  is a scalar random variable which is independent of  $Z$  following a left truncated standard normal at zero.

When these conditions are satisfied we have  $X$  following a one-factor skew-normal distribution and write:  $X \sim SN(\Delta, b, \Sigma)$ .

**Lemma 1:** *If the random vector of security returns follows a one-factor skew-normal distribution,  $X \sim SN(\Delta, b, \Sigma)$ , then for every risk averse investor with CARA utility  $u(w) = -\exp(-\alpha w)$  and initial wealth  $w_0$ , the optimal portfolio must satisfy:*

$$a = a_2 + \frac{e' \Sigma^{-1} \Delta}{k} (a_1 - a_2) + \frac{e' \Sigma^{-1} b (h(ka'b) - ka'b)}{k} (a_3 - a_2)$$

where  $a_1 = \frac{\Sigma^{-1} \Delta}{e' \Sigma^{-1} \Delta}$ ,  $a_2 = \frac{\Sigma^{-1} e}{e' \Sigma^{-1} e}$ ,  $a_3 = \frac{\Sigma^{-1} b}{e' \Sigma^{-1} b}$ ,  $k = \alpha w_0$  and the function  $h(ka'b)$  is the hazard function of a standard normal.

Proof of the Lemma 1: see appendix 1

Notice that the mean-variance-skewness efficient set, when assets follow  $X \sim SN(\Delta, b, \Sigma)$ , is generated by the three funds  $a_1$ ,  $a_2$  and  $a_3$ .

**Theorem 3:** *Let  $X \in R^n$  such that  $X \sim SN(\Delta, b, \Sigma)$  where the matrix  $\Sigma$  is non-singular, then the necessary and sufficient conditions for  $a^u \prec_m e^i$  (for non negative (positive) diversification),  $\forall u \in U$  are:*

- i)  $\Delta = k_1 e$  for some  $k_1 \in R$
- ii)  $b = k_2 e$  for some  $k_2 \in R$
- iii)  $\Sigma^{-1}e \geq (>)0$

Proof of theorem 3:

If the claim were to hold for all risk averse investors it must hold for CARA investors. According to the previous lemma, for CARA utility, the optimal portfolio must satisfy:

$$a = a_2 + \frac{e' \Sigma^{-1} \Delta}{k} (a_1 - a_2) + \frac{e' \Sigma^{-1} b (h(ka'b) - ka'b)}{k} (a_3 - a_2)$$

$$\text{where } a_1 = \frac{\Sigma^{-1} \Delta}{e' \Sigma^{-1} \Delta}, \quad a_2 = \frac{\Sigma^{-1} e}{e' \Sigma^{-1} e}, \quad a_3 = \frac{\Sigma^{-1} b}{e' \Sigma^{-1} b}$$

1/ If  $\Delta = k_1 e$  then  $a_1 = a_2$  and similarly if  $b = k_2 e$  then  $a_3 = a_2$ , this implies  $a = a_2$ .

Since  $\Sigma^{-1}e \geq (>)0$  by assumption and  $e' \Sigma^{-1}e > 0$  then  $a \geq (>)0$

2/ Suppose that  $a \geq 0 \quad \forall k$  then  $0 \leq a_j \leq 1 \quad \forall j$  since  $a'e = 1$

hence  $a'b$  must be bounded  $\forall k$  thus as  $k \rightarrow 0$  we have  $h(ka'b) \rightarrow h(0) = \sqrt{\frac{2}{\pi}}$

Consider the arbitrage portfolio  $\eta = e' \Sigma^{-1} \Delta (a_1 - a_2) + e' \Sigma^{-1} b h(\varepsilon) (a_3 - a_2)$

We show that  $\eta = 0$ , for suppose  $\eta \neq 0$ , then there must exist  $j \in N$  such that  $\eta_j < 0$

Since for arbitrarily small  $k$ ,  $\eta_j < 0$  implies  $a_j < 0$  we must have  $\eta = 0$ .

But  $\eta$  must be 0 for different values of  $\varepsilon$ , this can be satisfied only if  $a_1 = a_2 = a_3$

We conclude that  $\Delta = k_1 e$  and  $b = k_2 e$  hence  $a = a_2$  which must be nonnegative by assumption hence  $\Sigma^{-1}e \geq 0$ . End of proof

**Remark:** In the previous theorem and lemma, the distributional assumption about the scalar random variable  $y$  is not essential. As long as the specified distribution is skewed the results still hold. For example, we might adopt, as in Simaan (1993b) the Pearson type III class of distributions without altering the results.

Following the recent development of Sahu et al. (2003), we consider an alternative model which is not implied by the generating process

specified by Simaan (1993a). Instead of skewness being generated by a single factor for the whole economy upon which different securities have different loadings, as in Simaan, here skewness is generated by independent different factors for different securities. Consider the model:

$$X = \Delta + BY + Z$$

such that  $Z \sim N(0, \Sigma)$  with  $\Sigma$  non singular,  $B$  is a diagonal matrix with  $i^{th}$  component  $b_i$  and  $Y$  is a vector of  $n$  independent random variables which are independent of  $Z$  and each following a left truncated standard normal at zero.

When these conditions are satisfied we say that  $X$  follows a multi-factor skew-normal distribution and write:  $X \sim SN(\Delta, B, \Sigma)$ .

**Lemma 2:** *If the random vector of security returns follows a multi-factor skew-normal distribution,  $X \sim SN(\Delta, B, \Sigma)$ , then for every risk averse investor with CARA utility  $u(w) = -\exp(-\alpha w)$  and initial wealth  $w_0$ , the optimal portfolio must satisfy:*

$$a = a_2 + \frac{e'(\Sigma + B^2)^{-1} \Delta}{k} (a_1 - a_2) + \frac{e'(\Sigma + B^2)^{-1} V}{k} (a_3 - a_2)$$

$$\text{where } a_1 = \frac{(\Sigma + B^2)^{-1} \Delta}{e'(\Sigma + B^2)^{-1} \Delta}, \quad a_2 = \frac{(\Sigma + B^2)^{-1} e}{e'(\Sigma + B^2)^{-1} e}, \quad a_3 = \frac{(\Sigma + B^2)^{-1} V}{e'(\Sigma + B^2)^{-1} V} \quad \text{and}$$

$$k = \alpha w_0$$

The vector  $V$  has the  $i^{th}$  component given by  $V_i = b_i h(ka_i b_i)$  and  $h(ka_i b_i)$  is the hazard function of a standard normal.

Proof of the Lemma 2: see appendix 1

Unlike the previous model, we do not have a three-fund separation here since  $a_3$  is dependent on investors preferences.

**Theorem 4:** *Let  $X \in R^n$  such that  $X \sim SN(\Delta, B, \Sigma)$  where the matrix  $\Sigma$  is non-singular, then  $a^u \prec_m e^i \quad \forall u \in U$  if and only if:*

$$I) \quad \Delta = k_1 e \quad \text{for some } k_1 \in R$$

$$ii) \quad B = k_2 I \quad \text{for some } k_2 \in R$$

iii)  $(\Sigma + k_2^2 I)^{-1} e = k_3 e$  for some  $k_3 \in R_+$

In this case,  $\forall u \in U$ , the optimal solution is  $a = \frac{1}{n} e$

Proof of theorem 4:

For CARA utility, the optimal portfolio must satisfy:

$$a = a_2 + \frac{e'(\Sigma + B^2)^{-1} \Delta}{k} (a_1 - a_2) + \frac{e'(\Sigma + B^2)^{-1} V}{k} (a_3 - a_2) \text{ with } k = \alpha w_0$$

$$\text{where } a_1 = \frac{(\Sigma + B^2)^{-1} \Delta}{e'(\Sigma + B^2)^{-1} \Delta}, \quad a_2 = \frac{(\Sigma + B^2)^{-1} e}{e'(\Sigma + B^2)^{-1} e}, \quad a_3 = \frac{(\Sigma + B^2)^{-1} V}{e'(\Sigma + B^2)^{-1} V} \text{ and the}$$

vector  $V$  has the  $i^{th}$  component given by  $V_i = b_i h(ka_i b_i)$  with  $h(ka_i b_i)$  denoting the hazard function of a standard normal.

1/ If  $\Delta = k_1 e$  and  $(\Sigma + k_2^2 I)^{-1} e = k_3 e$  then  $a_1 = a_2 = \frac{1}{n} e$ , but since  $b_i = k_2$  then  $V_i = k_2 h(kk_2 a_i)$

and  $a$  must satisfy  $a = \frac{1}{n} e + \frac{k_3 e' V}{k} \left( a_3 - \frac{1}{n} e \right)$  with a solution given by

$$a = \frac{1}{n} e \quad \forall k.$$

2/ The argument here is essentially the same as the one used in the proof of the previous theorem. It suffices to note that as  $k$  gets arbitrarily small

$V_i = b_i h(ka_i b_i)$  gets arbitrarily close to  $b_i \sqrt{\frac{2}{\pi}}$  which implies that  $b_i = k_2 \quad \forall i$

for some  $k_2$                       End of proof

Notice that for all risk-averse investors to hold non negative (positive) amounts of each asset, the distributions, considered above, have to exhibit a one-fund separation.

Ross (1978) characterized the one-fund separating (1fs) distributions of asset returns. According to Ross (see p.263), a necessary condition for 1fs is that all assets have the same mean, furthermore, if assets have finite variances then the separating fund is simply the minimum variance portfolio.

We conjecture that for all risk-averse investors to hold non negative (positive) amounts of each asset with finite variances, the necessary and sufficient conditions are that the distribution, with the non-singular covariance matrix  $\Sigma$ , exhibits a one-fund separation and  $\Sigma^{-1}e \geq (>)0$ . But at this stage, we might state the weaker following result.

**Theorem 5:** *Let  $X \in R^n$  following a lfs distribution with all assets having finite variances and the covariance matrix  $\Sigma$  being non-singular, if  $\Sigma^{-1}e \geq (>)0$  then  $a^u \prec_m e^i \quad \forall u \in U$ . Thus every risk averse investor holds non negative (positive) amounts of each asset.*

A general useful notion of negative interdependence between random variables, called negative association, and the related concept of M-matrices can be found in Karlin & Rinott, (1982), Marshall and Olkin (1979) and references therein.

**Definition:** Let  $N = \{1, 2, \dots, n\}$  and define  $A_1, A_2 \subset N$  such that  $A_1 \cap A_2 = \emptyset$ , the set of random variables  $\{X_i, i \in N\}$  is said to be negatively associated if and only if  $\text{cov}(f(X_i, i \in A_1), g(X_i, i \in A_2)) \leq 0$  for all nondecreasing functions  $f, g$  and all disjoint sets  $A_1, A_2 \in N$ .

It is quite clear that negative association, as defined above, is sufficient for  $\text{cov}(X_i, X_j) \leq 0 \quad \forall i \neq j$ .

Note also that the definition, given above, is closely related to Epstein and Tanny's notion of "correlation increasing transformation". (see theorem 8 in Epstein and Tanny (1980)).

**Definition:** A square matrix  $\Omega$  is called an M-matrix if  $\Omega = \delta I - B$  where  $I$  denotes the identity matrix,  $B \geq 0$  and  $\delta$  is positive and exceeds the absolute value of every characteristic root of  $B$ .

Fan (1964), cited by Marshall and Olkin (1979) and Karlin & Rinott (1982), showed that a necessary and sufficient condition for a square matrix  $\Omega = (\omega_{ij})$  with  $\omega_{ij} \leq 0 \quad i \neq j$  to be an M-matrix is that  $\Omega$  is nonsingular and  $\Omega^{-1} = (\omega^{ij})$  with  $\omega^{ij} \geq 0, \forall i, j$ . In the case  $\Omega$  is an M-matrix and  $X \sim N(ke, \Omega)$  then according to Jogdeo and Proschan (1981), cited by Karlin & Rinott (1982), the random variables are negatively associated.

From the discussion above, it becomes clear that the inverse of an M-matrix has necessarily non negative row sums; we have, thus, the following result:

**Theorem 6:** Let  $X \in R^n$  following a lfs distribution with all assets having finite variances if the covariance matrix  $\Sigma$  is an M-matrix then  $a^u \prec_m e^i$   $\forall u \in U$ .

Proof: Since  $\Sigma$  is an M-matrix by Fan's theorem the inverse exists and  $\Sigma^{-1} \geq 0$  hence  $\Sigma^{-1}e \geq 0$ . By theorem 3 every risk adverse investor hold non negative amounts of each asset...

We turn now to the condition of exchangeability of asset returns distribution (see Samuelson (1967b)). While two identical assets are exchangeable, this is not necessarily true for  $n > 2$  assets, since symmetry of the joint distribution requires more restrictions due to the nature of the statistical interdependence between these assets. In this sense, for total diversification to hold for all investors, assets need to be more than identical individually, they must be identical in their relations to each other as well; hence the joint distribution must be symmetric. If assets are exchangeable, then "there is no sufficient reason to distinguish" between them hence "there can be no distinction" and the proportions invested in these assets should be equal. This is an example of what Polya (1967), cited by Marchall & Olkin (1979), following Leibniz, calls "the principle of nonsufficient reason".

**Definition:**  $X \in R^n$  is said to be exchangeable if  $X$  and  $X\Pi_i$  have the same distribution for all permutation matrices  $\Pi_i$   $i \in \{1, 2, \dots, n\}$ .

**Theorem 7: The theorem of total diversification.**

If  $X \in R^n$  is exchangeable and  $u \in U$ , the function  $\psi(a) = E(u(a'X))$  is symmetric concave and hence Schur-concave. That is  $a \prec_m b \Rightarrow \psi(a) > \psi(b)$

hence the optimal solution to the portfolio problem is  $a^* = \frac{1}{n}e$

Proof: Apply proposition B.2.c. page 288 of A. Marshall & I. Olkin

Notice that, when  $X \in R^n$  exchangeable, not only total diversification is optimal, but more importantly the expected utility function is increasing with greater diversification as defined above.

**Remark:**

When  $X \in R^n$  is exchangeable then  $E(X) = \Delta = ke$  for some  $k \in R$  and  $V(X) = \Sigma = \sigma^2((1 - \rho)I + \rho J)$  where  $\rho \in \left[-\frac{1}{n-1}, +1\right]$ ,  $I$  is the identity matrix and  $J = ee'$ . In this case we say that  $\Sigma$  has exchangeable structure. The matrix  $P = (1 - \rho)I + \rho J$  is the equicorrelation matrix, it is non singular (see Mardia, Kent & Bibby (1979) p. 461) only if  $\rho \in \left[-\frac{1}{n-1}, +1\right]$  in this case  $P^{-1} = (1 - \rho)^{-1} \left[ I - \rho(1 + (n-1)\rho)^{-1} J \right]$

The case of exchangeability suggests the following questions: Does more similarity (as measured by the above preorder relation) between relevant parameters (e.g. between expected asset returns or between variances) improve diversification?

The other question is related to the role played by the correlation between assets. Specifically, under what conditions more negative correlation improve the case of diversification?

The following propositions give cases in which the questions, raised above, have an unambiguous answer.

**Proposition 1:** Let  $X \sim SN(\Delta, k_1 e, \Sigma_1)$  and  $Y \sim SN(\mu, k_2 e, \Sigma_2)$  such that  $\Sigma_1 = \sigma_1^2 P_1$  and  $\Sigma_2 = \sigma_2^2 P_2$  both have exchangeable structures with correlation coefficients  $\rho_1, \rho_2$  respectively then,  $a_X \prec_m a_Y$  for all constant absolute risk-averse investors provided that  $\hat{\Delta} \prec_m \hat{\mu}$  and

$$\frac{e' \Delta}{(1 - \rho_1) \sigma_1^2} \leq \frac{e' \mu}{(1 - \rho_2) \sigma_2^2} \text{ where } \hat{\Delta} = \frac{\Delta}{e' \Delta}$$

Proof of proposition 1:

Assume CARA utility function with  $\frac{-u''(y)}{u'(y)} = \alpha$  and  $\alpha \in R_+$ , then the

optimal solutions is given by  $a = a_2 + \frac{1}{\alpha w} \eta$ , where  $\eta = e' \Sigma^{-1} \Delta (a_1 - a_2)$ ,

$$a_2 = \frac{\Sigma^{-1} e}{e' \Sigma^{-1} e} \text{ and } a_1 = \frac{\Sigma^{-1} \Delta}{e' \Sigma^{-1} \Delta}.$$

Furthermore, if we assume that  $\Sigma$  has exchangeable structure then  $a_2 = \frac{1}{n}e$

$$\text{and } a_1 = \frac{P^{-1}\Delta}{e'P^{-1}\Delta}.$$

On the other hand,  $P = (1 - \rho)I + \rho J$  is an equicorrelation matrix by assumption; its inverse is given by:  $P^{-1} = (1 - \rho)^{-1} [I - \rho(1 + (n - 1)\rho)^{-1} J]$ .

Hence we have:

$$\begin{aligned} P^{-1}\Delta &= (1 - \rho)^{-1} e'\Delta [\hat{\Delta} - \rho(1 + (n - 1)\rho)^{-1} e] \\ e'P^{-1}\Delta &= (1 - \rho)^{-1} e'\Delta [1 - n\rho(1 + (n - 1)\rho)^{-1}] \\ a_1 &= \frac{P^{-1}\Delta}{e'P^{-1}\Delta} = \left( \frac{1}{1 - n\rho(1 + (n - 1)\rho)^{-1}} \right) \hat{\Delta} + \left( \frac{-n\rho(1 + (n - 1)\rho)^{-1}}{1 - n\rho(1 + (n - 1)\rho)^{-1}} \right) \left( \frac{1}{n}e \right) \\ e'\Sigma^{-1}\Delta &= \sigma^{-2} e'P^{-1}\Delta = \sigma^{-2} (1 - \rho)^{-1} e'\Delta [1 - n\rho(1 + (n - 1)\rho)^{-1}] \\ a &= \frac{1}{n}e + \frac{e'\Delta}{\alpha w(1 - \rho)\sigma^2} \left( \hat{\Delta} - \frac{1}{n}e \right) \end{aligned}$$

Notice that the function  $\theta = \frac{e'\Delta}{\alpha w(1 - \rho)\sigma^2}$  is increasing in  $e'\Delta$  and  $\rho$  and decreasing  $\sigma^2$ .

Now since  $\hat{\Delta} \prec_m \hat{\mu}$  on  $\Lambda$  we have  $\left( \hat{\Delta} - \frac{1}{n}e \right) \prec_m \left( \hat{\mu} - \frac{1}{n}e \right)$  on  $\Lambda_0$ . On the

other hand, since  $\frac{e'\Delta}{((1 - \rho_1)\sigma_1^2)} \leq \frac{e'\mu}{(1 - \rho_2)\sigma_2^2}$  we must have

$$\frac{e'\Delta}{(1 - \rho_1)\sigma_1^2} \left( \hat{\Delta} - \frac{1}{n}e \right) \prec_m \frac{e'\mu}{(1 - \rho_2)\sigma_2^2} \left( \hat{\mu} - \frac{1}{n}e \right) \text{ on } \Lambda_0$$

We can conclude that  $a_X \prec_m a_Y$  End of proof.

**Proposition 2:** Let  $X \sim SN(k_1 e, b, \Sigma_1)$  and  $Y \sim SN(k_2 e, d, \Sigma_2)$  such that  $\Sigma_1 = \sigma_1^2 P_1$  and  $\Sigma_2 = \sigma_2^2 P_2$  both have exchangeable structures with correlation coefficients  $\rho_1, \rho_2$  respectively then,  $a_X \prec_m a_Y$  for all constant absolute risk-averse investors provided that  $b \prec_m d$  and

$$\frac{1}{((1 - \rho_1)\sigma_1^2)} \leq \frac{1}{(1 - \rho_2)\sigma_2^2}$$

Proof of proposition 2:

Assume CARA utility function with  $\frac{-w_0 u''(w_0)}{u'(w_0)} = k$  and  $X \sim SN(k_1 e, b, \Sigma)$ ,

by Lemma 1 the optimal solution must satisfy  $a = a_2 + \frac{1}{\alpha w} \eta$ , where

$$\eta = e' \Sigma^{-1} b (a_3 - a_2), \quad a_2 = \frac{\Sigma^{-1} e}{e' \Sigma^{-1} e} \quad \text{and} \quad a_3 = \frac{\Sigma^{-1} b}{e' \Sigma^{-1} b}.$$

When  $\Sigma$  has exchangeable structure then  $a_2 = \frac{1}{n} e$ ,  $a_3 = \frac{P^{-1} b}{e' P^{-1} b}$  and we

$$\text{have: } a = \frac{1}{n} e + \frac{e' b (h(ka'b) - ka'b)}{k(1-\rho)\sigma^2} \left( \hat{b} - \frac{1}{n} e \right) = \frac{1}{n} e + \frac{h(ka'b) - ka'b}{k(1-\rho)\sigma^2} (b - \bar{b} e)$$

Premultiplying by  $b'$ ,  $a'b$  must satisfy the condition:

$$a'b - \bar{b} = \frac{h(ka'b) - ka'b}{k(1-\rho)\sigma^2} (b'b - n\bar{b}^2).$$

We write this condition in the form:  $g(z) = h(z) - z = \frac{(1-\rho)\sigma^2(z - k\bar{b})}{nS_b^2}$

where  $S_b^2$  is the variance of the vector  $b$ .

According to Marshall & Olkin (see page 493), the hazard function  $h(z)$  of the normal is increasing which implies that  $g(z) > 0$  since  $h'(z) = h(z)g(z)$

Furthermore, notice that  $g(0) = \sqrt{\frac{2}{\pi}}$ ,  $\lim_{z \rightarrow +\infty} g(z) = 0$  as  $z \rightarrow +\infty$  and

$\lim_{z \rightarrow -\infty} g(z) = +\infty$  as  $z \rightarrow -\infty$ .

Consider  $X$  and  $Y$  with the two conditions:

$$g(z) = h(z) - z = \frac{(1-\rho_1)\sigma_1^2(z - k\bar{b})}{nS_b^2} \quad (1)$$

$$g(z) = h(z) - z = \frac{(1-\rho_2)\sigma_2^2(z - k\bar{d})}{nS_d^2} \quad (2)$$

Now, according to Marshall & Olkin (see page 71), the function  $S_b^2 = f(b)$  is schur-convex. Hence  $b \prec_m d$  implies  $S_b^2 < S_d^2$ . Also from

$$(1-\rho_1)\sigma_1^2 \geq (1-\rho_2)\sigma_2^2 \quad \text{we have} \quad \frac{(1-\rho_1)\sigma_1^2(z - k\bar{b})}{nS_b^2} > \frac{(1-\rho_2)\sigma_2^2(z - k\bar{d})}{nS_d^2} > 0$$

for  $z > k\bar{b} = k\bar{d}$

We have established that the solutions to (1) and (2) must satisfy  $z_1 < z_2$ .

On the other hand,  $a_X = \frac{1}{n}e + \frac{g(z_1)}{k(1-\rho_1)\sigma_1^2}(b - \bar{b}e)$  and

$$a_Y = \frac{1}{n}e + \frac{g(z_2)}{k(1-\rho_2)\sigma_2^2}(d - \bar{d}e)$$

Since  $b \prec_m d$  we can conclude that  $a_X \prec_m a_Y$  End of proof.

**Proposition 3:** Let  $X$  and  $Y$  two multivariate lfs separating distribution with the finite covariance matrices  $\Sigma = D(\sigma_i)P_1D(\sigma_i)$ ,  $\Omega = D(\sigma_i)P_2D(\sigma_i)$  where  $P_1, P_2$  have equicorrelation structure with correlation coefficients  $\rho_1, \rho_2$  respectively. if  $\rho_1 < \rho_2$  then all risk-averse investors diversify more under  $X$ .

Proof of proposition 3:

The optimal solutions  $a_X$  and  $a_Y$  must satisfy:  $a_X = \frac{\Sigma^{-1}e}{e'\Sigma^{-1}e}$

and  $a_Y = \frac{\Omega^{-1}e}{e'\Omega^{-1}e}$ , where  $\Sigma = D(\sigma_i)P_1D(\sigma_i)$  and  $\Omega = D(\sigma_i)P_2D(\sigma_i)$

Let  $d(\sigma_i^{-1}) = D(\sigma_i^{-1})e$  and  $\hat{d}(\sigma_i^{-1}) = \frac{1}{e'd(\sigma_i^{-1})}d(\sigma_i^{-1})$ , we then have:

$$a_X = \frac{\Sigma^{-1}e}{e'\Sigma^{-1}e} = \frac{D(\sigma_i^{-1})P^{-1}d(\sigma_i^{-1})}{d'(\sigma_i^{-1})P^{-1}d(\sigma_i^{-1})}$$

The matrix  $P = (1-\rho)I + \rho J$  is an equicorrelation matrix by assumption; its inverse is given by:  $P^{-1} = (1-\rho)^{-1}[I - \rho(1+(n-1)\rho)^{-1}J]$ .

Hence  $D(\sigma_i^{-1})P^{-1}d(\sigma_i^{-1}) = (1-\rho)^{-1}[d(\sigma_i^{-2}) - \rho(1+(n-1)\rho)^{-1}e'd(\sigma_i^{-1})d(\sigma_i^{-1})]$

We conclude that  $a_X = \theta(\rho, \sigma_i)\hat{d}(\sigma_i^{-2}) + (1-\theta(\rho, \sigma_i))\hat{d}(\sigma_i^{-1})$  (1)

Where  $\theta(\rho, \sigma_i) = \frac{e'd(\sigma_i^{-2})}{e'd(\sigma_i^{-2}) - \rho(1 + (n-1)\rho)^{-1}(e'd(\sigma_i^{-1}))^2}$ , notice that the function  $\theta(\rho, \sigma_i)$  is

increasing in  $\rho$  that is  $\rho_1 < \rho_2 \Rightarrow \theta(\rho_1, \sigma_i) < \theta(\rho_2, \sigma_i)$  (2).

On the other hand, according to A. Marshall & I. Olkin (See proposition B.2.b. page 130) if  $d_i > 0, i=1,2,\dots,n$  and  $0 < r \leq s$  then

$$\frac{(d_1^r, d_2^r, \dots, d_n^r)}{\sum d_i^r} \prec_m \frac{(d_1^s, d_2^s, \dots, d_n^s)}{\sum d_i^s}$$

$$\therefore \hat{d}(\sigma_i^{-1}) \prec_m \hat{d}(\sigma_i^{-2}) \quad (3)$$

From (1), (2) and (3) we conclude that  $a_X \prec_m a_Y$ . End of proof.

#### Remark:

From the previous propositions a direction towards more negative correlation, as expected, imply more diversification even when the distribution is not elliptical and hence not normal.

From proposition 3, notice that the optimal solution for all risk averse investor is a combination of the two funds  $\hat{d}(\sigma_i^{-1})$  and  $\hat{d}(\sigma_i^{-2})$ . Recall that

$$\hat{d}(\sigma_i^{-1}) \prec_m \hat{d}(\sigma_i^{-2})$$

Since  $\theta(0, \sigma_i) = 1$  and  $\theta(\rho, \sigma_i) \rightarrow 0$  as  $\rho \rightarrow \frac{-1}{n-1}$ , we see that when assets are uncorrelated then the optimal solution is  $\hat{d}(\sigma_i^{-2})$ . On the other hand, when  $\rho$  is negative the optimal strategy is to hold both  $\hat{d}(\sigma_i^{-1})$  and  $\hat{d}(\sigma_i^{-2})$  long and short sales are not optimal. In the limit, as  $\rho$  gets sufficiently small the fund  $\hat{d}(\sigma_i^{-1})$  is the optimal portfolio. Furthermore, as  $\rho$  gets large the fund  $\hat{d}(\sigma_i^{-1})$  is held short, when it gets sufficiently larger, short sales become optimal.

**Proposition 4:** Let  $X$  and  $Y$  two multivariate lfs separating distribution with the finite covariance matrices  $\Sigma = D(\sigma_i)PD(\sigma_i)$ ,  $\Omega = D(\delta_i)PD(\delta_i)$  and  $P$  has equicorrelation structure and assume, without loss of generality, that  $\sigma_i$  and  $\delta_i$  are decreasingly ordered, if  $\frac{\sigma_i}{\delta_i}$  is decreasing in  $i=1,2,\dots,n$

and  $\rho \leq 0$  then  $a_X \prec_m a_Y$  and all risk-averse investors diversify more under  $X$ .

Proof of proposition 4:

From the proof of proposition 2 the optimal solutions  $a_X$  and  $a_Y$  must satisfy:

$$\begin{aligned} a_X &= \theta(\rho, \sigma_i) \hat{d}(\sigma_i^{-2}) + (1 - \theta(\rho, \sigma_i)) \hat{d}(\sigma_i^{-1}) \\ a_Y &= \theta(\rho, \delta_i) \hat{d}(\delta_i^{-2}) + (1 - \theta(\rho, \delta_i)) \hat{d}(\delta_i^{-1}) \end{aligned} \quad (1)$$

$$\text{Where } \theta(\rho, \sigma_i) = \frac{e' d(\sigma_i^{-2})}{e' d(\sigma_i^{-2}) - \rho(1 + (n-1)\rho)^{-1} (e' d(\sigma_i^{-1}))^2},$$

Recall that we have  $\hat{d}(\sigma_i^{-1}) \prec_m \hat{d}(\sigma_i^{-2})$  and similarly  $\hat{d}(\delta_i^{-1}) \prec_m \hat{d}(\delta_i^{-2})$  (2)

According to Marshall & Olkin (see proposition B.1.b p.129) let  $d_i > 0, i = 1, 2, \dots, n$  and  $b_i > 0, i = 1, 2, \dots, n$  if  $\frac{b_i}{d_i}$  is decreasing in  $i$  then

$$\begin{aligned} \frac{(d_1, d_2, \dots, d_n)}{\sum d_i} &\prec_m \frac{(b_1, b_2, \dots, b_n)}{\sum b_i} \quad \text{and} \\ \frac{(d_1^2, d_2^2, \dots, d_n^2)}{\sum d_i^2} &\prec_m \frac{(b_1^2, b_2^2, \dots, b_n^2)}{\sum b_i^2} \end{aligned}$$

If  $\frac{\sigma_i}{\delta_i}$  is decreasing in  $i = 1, 2, \dots, n$  then  $\hat{d}(\sigma_i^{-1}) \prec_m \hat{d}(\delta_i^{-1})$  and

$$\hat{d}(\sigma_i^{-2}) \prec_m \hat{d}(\delta_i^{-2}) \quad \text{where } d(\sigma_i^{-1}) = D(\sigma_i^{-1})e \text{ and } \hat{d}(\sigma_i^{-1}) = \frac{1}{e' d(\sigma_i^{-1})} d(\sigma_i^{-1}).$$

(3)

Also, from Marshall & Olkin (see proposition B.3 P.130), the function

$$\frac{(e' d(\sigma_i^{-1}))^2}{e' d(\sigma_i^{-2})} \text{ is schur-concave (4)}$$

From (2) and (3) and (4) we conclude that  $a_X \prec_m a_Y$ . End of proof

### Remark:

Notice that, in the previous propositions,  $Y$  is not a mutual fund transformation of  $X$ . Hence the mutual funds transformation is not necessary for more diversification.

## Conclusion:

An important motivation for the theory of portfolio choice is the phenomenon of diversification. The theory has developed from a need to rationalize diversification as an optimal investment strategy and from a need to derive the financial consequences of this strategy. As explained by Scheffman, the interest in knowing whether investors would tend to diversify (hold all assets long), to concentrate (hold only one asset) or take short positions has important implications for the "size" and "depth" of the capital markets and the effects of various changes on these markets.

In this article, we use the notion of majorization between two vectors to propose a general definition of « more diversified » in portfolio problems. It is argued that this definition is, on one hand, very intuitive. Portfolio  $a$  is more diversified than portfolio  $b$  means that the sum of the  $k$  largest proportions of portfolio  $a$  is not larger than that of portfolio  $b$  for all  $k$ . On the other hand, it is appealing because it is independent of investors' preferences as well as of returns distributions. Furthermore, it is potentially useful in providing a unified conceptual framework to the theory of diversification in portfolio selection problems.

To relate this definition to the optimal investment strategy and to characterize the relation between asset returns distributions that make all risk adverse investors diversify more under one distribution than under the other, we use both the general expected utility model for arbitrary distributions as well as standard parametric distributions to address the question of what it is that is diversified and what the desirable degree of diversification is, emphasizing the role of different parameters (mean, variance, skewness,...) in determining the degree of diversification. Some specific results are obtained. Perhaps, the most important result is that of the mutual fund transformation. It is shown that if the random vector of returns  $Y$  is a mutual fund transformation of  $X$  then every risk averse investor diversify more under  $X$ . This transformation insures that the difference between  $X$  and  $Y$  that causes every risk averse investor to diversify more under  $X$  is not a result of a change in the overall return-risk choices but rather a result of a redistribution of these choices. Furthermore, for elliptical and skew-normal distributions, necessary and sufficient conditions are given to insure that short sales are not the optimal investment strategy for all risk-averse investors. We also show, under certain distributional assumptions, that majorization between vectors of relevant parameters of two asset returns distributions is sufficient for more diversification by risk averse investors, in this case the difference between  $X$  and  $Y$  that causes every

risk averse investor to diversify more under  $X$  may be a result of a change in the overall return-risk choices. As to the role played by the correlation between assets, a direction towards more negative correlation, under some conditions, imply more diversification even when the distribution is not elliptical and hence not normal.

## APPENDIX: The portfolio selection problem when asset returns follows skew-normal distribution

### I. One-factor skew-normal distributions

Consider the model:

$$X = \Delta + by + Z$$

such that  $Z \sim N(0, \Sigma)$  with  $\Sigma$  non singular and  $y$  is a scalar random variable which is independent of  $Z$  following a left truncated standard normal at zero.

When these conditions are satisfied we say that  $X$  follows a skew-normal distribution and write:  $X \sim SN(\Delta, b, \Sigma)$ .

When  $y$  follows a left truncated standard normal at zero, then its density is given by:

$$f(y) = \sqrt{\frac{2}{\pi}} \exp\left(-\frac{y^2}{2}\right) \text{ where } y > 0$$

The moment generating function of  $y$  is given by:

$$M_y(t) = E(\exp(ty)) = 2 \exp\left(\frac{t^2}{2}\right) \phi(t)$$

where  $\phi(t)$  is the cumulative distribution of a standard normal.

To derive the moment generating function of  $X$  we write:

$$M_X(t) = E(\exp(t'X)) = E(\exp(t'\Delta + t'by + t'Z)) = E_y E_{X/y}(\exp(t'\Delta + t'by + t'Z))$$

but  $X/y \sim N(\Delta + by, \Sigma)$  hence

$$M_X(t) = E_y \left( \exp\left(t'\Delta + t'by + \frac{t'\Sigma t}{2}\right) \right)$$

$$M_X(t) = \exp\left(t'\Delta + \frac{t'\Sigma t}{2}\right) E_y(\exp(t'by)) = 2 \exp\left(t'\Delta + \frac{t'\Sigma t}{2}\right) \exp\left(\frac{(t'b)^2}{2}\right) \phi(t'b)$$

$$M_X(t) = 2 \exp\left(t'\Delta + \frac{t'(\Sigma + bb')t}{2}\right) \phi(t'b)$$

To solve the portfolio choice problem for a risk averse investor with CARA utility  $u(w) = -\exp(-\alpha w)$  and initial wealth  $w_0$  and assuming that  $X \sim SN(\Delta, b, \Sigma)$

$$\max E(u(w_0 a'X)) = \min E \exp(-\alpha w_0 a'X) \quad \text{such that } a'e = 1$$

Set  $k = \alpha w_0$  and observe that  $\ln(\cdot)$  is a monotone increasing transformation, the two problems  $\min E \exp(-\alpha w_0 a'X)$  and  $\min \ln(E \exp(-\alpha w_0 a'X))$  are equivalent

Define the lagrange function

$$L = \ln(2) - ka'\Delta + \frac{k^2 a'(\Sigma + bb')a}{2} + \ln(\phi(-ka'b)) + \theta(1 - a'e)$$

From the first order conditions of optimality we have

$$\frac{\partial L}{\partial a} = -k\Delta + k^2(\Sigma + bb')a - k \frac{\phi(-ka'b)}{\phi(-ka'b)} b - \theta e = 0 \quad \text{and } a'e = 1$$

Notice that the function  $h(ka'b) = \frac{\phi(-ka'b)}{\phi(-ka'b)} = \frac{\phi(ka'b)}{1 - \phi(ka'b)}$  is the hazard function of a standard normal.

premultiplying by  $\Sigma^{-1}$ , we get

$$k^2 a = k\Sigma^{-1}\Delta - k^2 a'b\Sigma^{-1}b + kh(ka'b)\Sigma^{-1}b + \theta \Sigma^{-1}e$$

premultiplying again by  $e'$ , we get

$$\theta = \frac{-ke'\Sigma^{-1}\Delta + k^2(1 + e'\Sigma^{-1}bb'a) - kh(ka'b)e'\Sigma^{-1}b}{e'\Sigma^{-1}e}$$

rearranging and collecting terms we have

$$a = a_2 + \frac{e'\Sigma^{-1}\Delta}{k}(a_1 - a_2) + \frac{e'\Sigma^{-1}b(h(ka'b) - ka'b)}{k}(a_3 - a_2)$$

$$\text{where } a_1 = \frac{\Sigma^{-1}\Delta}{e'\Sigma^{-1}\Delta}, \quad a_2 = \frac{\Sigma^{-1}e}{e'\Sigma^{-1}e} \quad \text{and} \quad a_3 = \frac{\Sigma^{-1}b}{e'\Sigma^{-1}b}$$

## II. Multi-factor skew-normal distributions

Consider the model:

$$X = \Delta + BY + Z$$

such that  $Z \sim N(0, \Sigma)$  with  $\Sigma$  non singular,  $B$  is a diagonal matrix and  $Y$  is a vector of  $n$  independent random variables which are independent of  $Z$  and each following a left truncated standard normal at zero.

When these conditions are satisfied we say that  $X$  follows a multi-factor skew-normal distribution and write:  $X \sim SN(\Delta, B, \Sigma)$ .

From the moment generating function of a univariate left truncated standard normal at zero, given above, and by independence, the moment generating function of  $Y$  is given by:

$$M_Y(t) = E(\exp(t'Y)) = 2^n \exp\left(\frac{t't}{2}\right) \phi_n(t)$$

where  $\phi_n(t)$  is the multivariate cumulative standard normal.

To derive the moment generating function of  $X$  we write:

$$M_X(t) = E(\exp(t'X)) = E(\exp(t'\Delta + t'BY + t'Z)) = E_Y E_{X/Y}(\exp(t'\Delta + t'BY + t'Z))$$

but  $X/Y \sim N(\Delta + BY, \Sigma)$  hence

$$M_X(t) = E_Y \left( \exp \left( t'\Delta + t'BY + \frac{t'\Sigma t}{2} \right) \right)$$

$$M_X(t) = \exp \left( t'\Delta + \frac{t'\Sigma t}{2} \right) E_Y (\exp(t'BY)) = 2^n \exp \left( t'\Delta + \frac{t'\Sigma t}{2} \right) \exp \left( \frac{t'B^2 t}{2} \right) \phi_n(Bt)$$

$$M_X(t) = 2^n \exp \left( t'\Delta + \frac{t'(\Sigma + B^2)t}{2} \right) \phi_n(Bt)$$

To solve the portfolio choice problem for a risk averse investor with CARA utility and initial wealth  $w_0$  and assuming that  $X \sim SN(\Delta, B, \Sigma)$ , we set  $k = \alpha w_0$  and write:

$$\min \ln(E \exp(-ka'X)) \quad \text{such that } a'e = 1$$

Define the Lagrange function

$$L = n \ln(2) - ka'\Delta + \frac{k^2 a'(\Sigma + B^2)a}{2} + \ln(\phi_n(-kBa)) + \theta(1 - a'e)$$

From the first order conditions of optimality we have

$$\frac{\partial L}{\partial a} = -k\Delta + k^2(\Sigma + B^2)a - kV - \theta e = 0 \text{ and } a'e = 1$$

where the vector  $V$  has the  $i^{th}$  component given by  $V_i = b_i h(ka_i b_i)$ ,  $b_i$  is the  $i^{th}$  component of  $B$  and  $h(ka_i b_i)$  is the hazard function of a standard normal.

premultiplying by  $\frac{1}{k^2}(\Sigma + B^2)^{-1}$ , we get

$$a = \frac{1}{k}(\Sigma + B^2)^{-1}\Delta + \frac{1}{k}(\Sigma + B^2)^{-1}V + \theta(\Sigma + B^2)^{-1}e$$

premultiplying again by  $e'$ , we determine  $\theta$  and write:

$$a = a_2 + \frac{e'(\Sigma + B^2)^{-1}\Delta}{k}(a_1 - a_2) + \frac{e'(\Sigma + B^2)^{-1}V}{k}(a_3 - a_2)$$

$$\text{where } a_1 = \frac{(\Sigma + B^2)^{-1}\Delta}{e'(\Sigma + B^2)^{-1}\Delta}, \quad a_2 = \frac{(\Sigma + B^2)^{-1}e}{e'(\Sigma + B^2)^{-1}e} \text{ and } a_3 = \frac{(\Sigma + B^2)^{-1}V}{e'(\Sigma + B^2)^{-1}V}$$

### Bibliography:

- Adcock C. (2002)**, "Asset pricing and portfolio selection based on the multivariate skew-student distribution", *Technical Report, University of Sheffield, England*.
- Adcock C. & Shutes K. (1999)**, "Portfolio selection based on the multivariate skew-normal distribution", *Working Paper, University of Bath, England*.
- Arrow K. (1971)**, "Essays in the theory of risk bearing", *Markham*.
- Azzalini A. & Dalla Valle A. (1996)**, "The multivariate skew-normal distribution", *Biometrika*.
- Bawa V. (1975)**, "Optimal rules for ordering uncertain prospects", *Journal of Economic Theory*.
- Black F. (1972)**, "Capital market equilibrium with restricted borrowing", *Journal of Business*.
- Blattberg R. & N. Gonedes (1974)**, "A comparison of the stable and student distribution as statistical models for stock prices", *Journal of Business*.
- Broca D. (2002)**, "The distribution of Indian stock returns: A tale of two tails" *Decision*.

- Brumelle S. (1974)**, "When does diversification between two investments pay?" *Journal of Financial and Quantitative Analysis*.
- Chamberlain G. (1983)**, "A characterization of the distributions that imply mean-variance utility functions", *Journal of Economic Theory*.
- Cass D. & Stiglitz J. (1970)**, "The structure of investor preferences and asset returns, and separability in portfolio selection: A contribution to the pure theory of mutual funds", *Journal of Economic Theory*.
- Chung P., Johnson H. & Schill M. (2001)**, "Asset pricing when returns are non normal: Fama-French factors verses higher order systematic co-moments", *AGSM University of California Riverside*.
- Conine T. & Tamarkin M. (1981)**, "On diversification given asymmetry in returns", *Journal of Finance*.
- Davies L. & Ronning G. (1974)**, "Existence, uniqueness and continuity of portfolio choice", *Zeitschrift fur Nationaloconomie*.
- Duffie D. & Pan J. (1997)**, "An overview of value at risk", *Journal of derivatives*.
- Dybvig P. (1985)**, "Acknowledgment: Kinks on the mean-variance frontier", *Journal of Finance*.
- Epps T. (1981)**, "Necessary and sufficient conditions for the mean-variance portfolio model with constant risk aversion", *Journal of Financial and Quantitative Analysis*.
- Epstein L. & Tanny S. (1980)**, "Increased generalized correlation: a definition and some economic consequences", *Canadian Journal of Economics*.
- Fama E. (1965a)**, "The behavior of stock market prices", *Journal of Business*.
- Fama E. (1965b)**, "Portfolio analysis in a stable paretian market", *Management Science*.
- Fama E. & French k. (1995)**, "Size and book to market factors in earnings and returns", *Journal of Finance*.
- Hadar J. & Russel W. (1969)**, "Rules for ordering uncertain prospects", *American Economic Review*.
- Hadar J. & Russel W. (1971)**, "Stochastic dominance and diversification", *Journal of Economic Theory*.
- Hadar J. & Russel W. (1974)**, "Diversification of interdependent prospects", *Journal of Economic Theory*.
- Hanoch G. & Levy H. (1969)**, "The efficiency analysis of choices involving risk", *Review of Economic Studies*.
- Johnson N. & Kotz S. (1972)**, "Distributions in statistics: continuous multivariate distributions", *John Wiley & sons*.

- Jurczenko E. & Maillet B. (2001)**, "The three-moment CAPM: Foundations and an asset pricing models comparison in a unified framework", in: *Dunis, Christian et al. (Eds), Developments in Forecast Combination and portfolio Choice, John Wiley & sons.*
- Kallberg J. & Ziemba T. (1979)**, "On the robustness of the Arrow-Pratt risk aversion measure", *Economics Letters*.
- Kallberg J. & Ziemba T. (1983)**, "Comparison of alternative utility functions in portfolio selection problems", *Management Science*.
- Karlin S. & Rinott Y. (1982)**, "M-matrices as covariance matrices of multinomial distributions", working paper, *Stanford University*.
- Kelker D. (1970)**, "Distribution theory of spherical distributions and a location scale parameter generalization", *Sankhya*.
- Konno H. & Yamazaki H. (1991)**, "Mean absolute deviation portfolio optimization model and its applications to Tokyo stock market", *Management Science*.
- Kraus A. & Litzenberger R. (1976)**, "Skewness preference and the valuation of risky assets", *Journal of Finance*.
- Levy H. (1979)**, "Does diversification always pay?" *TIMS Studies in the Management Sciences*.
- Levy H. & Markowitz H. (1979)**, "Approximating expected utility by a function of the mean and the variance", *American Economic Review*.
- Lintner J. (1965)**, "The valuation of risky assets and the selection of risky investments in stock portfolios and capital budgets", *Review of Economics and Statistics*.
- Longin F. (1995)**, "Le choix de la loi de rentabilités d'actifs financiers: Les valeurs extrêmes peuvent aider", *Finance, Revue de l'Association Française de Finance*.
- Marchall A. & Olkin I. (1979)**, "Inequalities: theory of majorization and its applications", *Academic Press*.
- Mardia K., Kent T. & Bibby JM. (1979)**, "Multivariate analysis", *Academic Press*.
- Markowitz H. (1952)**, "Portfolio selection", *Journal of Finance*.
- Markowitz H. (1959)**, "Portfolio selection: efficient diversification of investments", *Wiley New York*.
- Merton R. (1971)**, "Optimum consumption and portfolio rules in a continuous time model", *Journal of Economic Theory*.
- Merton R. (1972)**, "An analytical derivation of the efficient portfolio frontier", *Journal of Financial and Quantitative Analysis*.
- Mandelbrot B. (1963)**, "The variation of certain speculative prices", *Journal of Business*.
- Mossin J. (1966)**, "Equilibrium in a capital asset market", *Econometrica*.

- Muirhead R. (1982)**, "Aspects of multivariate statistical theory", *John Wiley & sons*.
- Nachman D. (1979)**, "On the theory of risk aversion and the theory of risk", *Journal of Economic Theory*.
- Nachman D. (1982)**, "Preservation of "more risk aversion" under expectation", *Journal of Economic Theory*.
- Owen J. & Rabinovitch R. (1983)**, "On the class of elliptical distributions and their applications to the theory of portfolio choice", *Journal of Finance*.
- Ross S. (1978)**, "Mutual funds separation in financial theory-The separating distributions", *Journal of Economic Theory*.
- Polya G. (1967)**, "Inequalities and the principle of nonsufficient reason", *In Inequalities, Academic press, New York*.
- Rothschild M. & Stiglitz J. (1970)**, "Increasing risk I: a definition", *Journal of Economic Theory*.
- Rubinstein M. (1973)**, "The fundamental theorem of parameter-preference security valuation", *Journal of Financial and Quantitative Analysis*.
- Rubinstein M. (1976)**, "The valuation of uncertain income streams and the pricing of options", *Bell Journal of Economics*.
- Sahu S., Dey D. & Branco M. (2003)**, "A new class of multivariate skew distributions with applications to Bayesian regression models", *The Canadian Journal of Statistics*.
- Samuelson P. (1967a)**, "Efficient portfolio selection for Pareto-Levy investments", *Journal of Financial and Quantitative Analysis*.
- Samuelson P. (1967b)**, "General proof that diversification pays", *Journal of Financial and Quantitative Analysis*.
- Samuelson P. (1970)**, "The fundamental approximation theorem of portfolio analysis in terms of means, variances, and higher moments", *Review of Economic Studies*.
- Scheffman D. (1975)**, "A definition of generalized correlation and its application for portfolio analysis", *Economic Inquiry*.
- Scott R. & Horvath P. (1980)**, "On the direction of preference for moments of higher order than variance", *Journal of Finance*.
- Sharpe W. (1963)**, "A simplified model for portfolio analysis", *Management Science*.
- Sharpe W. (1964)**, "Capital asset prices: a theory of market equilibrium under conditions of risk", *Journal of Finance*.
- Simaan Y. (1993a)**, "Portfolio selection and asset pricing-Three parameter framework", *Management Science*.
- Simaan Y. (1993b)**, "What is the opportunity cost of mean-variance investment strategies? ", *Management Science*.

- Simaan Y. (1997)**, "Estimation risk in portfolio selection: The mean-variance model versus the mean absolute deviation model", *Management Science*.
- Tobin J. (1958)**, "Liquidity preference as behavior towards risk", *Review of Economic Studies*.
- Touati-Tliba M. (1983)**, "On diversification in portfolio selection problems", *PhD Proposal, GBA, New York University*.
- Touati-Tliba M. (2003)**, "On portfolio selection when asset returns are elliptical", *Revue des Sciences Commerciales et de Gestion, L.E.P.S.C.S.G., E.S.C*
- Wang J. (2001)**, "Generating daily changes in market variables using a multivariate mixture of normal distributions", *Proceedings of the 2001 Winter Simulation Conference*.
- Whitmore G. (1970)**, "Third order stochastic dominance", *American Economic Review*.