

Monotone Theory for Hadamard Fractional Boundary Value Problems

نظرية التزايد في مسائل القيم الحدية ذات المشتقة الكسرية لهادامارد.

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Abstract:

In this paper, we study the existence and uniqueness of a weak solution for a fractional differential problem involving the Hadamard fractional derivative, defined on a bounded interval. The initial conditions are assumed to be homogeneous, and the right-hand side is continuous. The analysis is carried out within an appropriate functional framework, and the main results are established by applying the Minty–Browder theorem, which serves as an effective tool for proving the solvability of equations of this type.

Key words: Fractional Differential Equations, Hadamard Fractional, Boundary Value Problem, monotone operator, Minty-Browder theorem.

Introduction:

The variational method is one of the powerful and significant tools in nonlinear analysis, and it plays a crucial role in the theory of critical points. The variational principle has important applications in several fields for instance, in generalized differential calculus and it serves as an effective approach in applied analysis, approximation theory, and physics in general. Numerous studies and papers have been published concerning the existence of solutions for differential equations with various types of fractional derivatives, mainly using the fixed-point method. However, research works or scientific papers that employ the variational approach through the critical point theory remain relatively limited compared to the former method [12].

In [4], the authors discuss existence and uniqueness of a weak solution of a fractional problem on the half-line via the Minty-Browder theorem.

$$\begin{cases} D_-^\alpha (D_+^\alpha u(t)) + u(t) = f(t, u(t)), & t \in (0, +\infty), \\ u(0) = u(\infty) = 0, \end{cases}$$

Where $\frac{1}{2} < \alpha < 1$ and $f : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function.

The aim of this paper is to study the existence and uniqueness of a solution to the fractional Hadamard boundary value problem posed on a bounded interval as follows:

$$\begin{cases} {}^H D_{b^-}^\alpha (t {}^H D_{a^+}^\alpha u(t)) = \lambda f(t, u(t)), & t \in [a, b], \\ {}^H I_{a^+}^{1-\alpha} u(a) = u(b) = 0, \end{cases} \quad (1.1)$$

Where $0 < a < b < \infty$, and $\frac{1}{2} < \alpha < 1$, $\lambda > 0$. The function $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

The operators ${}^H D_{a^+}^\alpha$ and ${}^H D_{b^-}^\alpha$ denote the left and right Hadamard fractional derivatives, respectively.

The analysis is carried out using the variational method and monotone operator theory.

1. Preliminaries :

Definition 1.1. Let $-\infty < a < b < +\infty$, $1 \leq p < +\infty$. The space $L^p(a, b)$ is the set of measurable functions u on (a, b) such that $\|u\|_{L^p[a, b]} < +\infty$, defined by

$$\|u\|_{L^p[a, b]} = \left(\int_a^b |u(t)|^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p < +\infty,$$

$$\text{And } \|u\|_{L^\infty[a, b]} = \text{ess sup}_{a \leq t \leq b} |u(t)|,$$

Here $\text{ess sup}_{a \leq t \leq b} |u(t)|$ represents the essential maximum of the function $|u(t)|$. The space $L^p(a, b)$ is a Banach space. In addition, $L^2[a, b]$ is a Hilbert space with respect to the scalar product $\langle f, g \rangle = \int_a^b f(t) \cdot g(t) dt$.

Definition 1.2. [10] Let $-\infty < a < b < +\infty$, $1 \leq p < +\infty$, and $c \in \mathbb{R}$. The space $X_c^p(a, p)$ is the set of measurable functions u from (a, b) to $\mathbb{R}$ such that $\|u\|_{X_c^p[a, p]} < +\infty$.

It is a Banach space respect to the norm $\|u\|_{X_c^p[a, p]} = \left(\int_a^b |t^c u(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}}, \quad 1 \leq p < +\infty$

$$\text{And } \|u\|_{X_c^\infty[a, p]} = \text{ess sup}_{a \leq t \leq b} [t^c |u(t)|].$$

In particular, when $c = \frac{1}{p}$, the space $X_c^p(a, p)$ coincides with space

$$L^p(a, p) : X_{\frac{1}{p}}^p(a, b) = L^p(a, p).$$

Proposition 1.1 [10] Let $p > 1, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then for $u \in X_c^p(a, b)$ and $v \in X_c^q(a, b)$ we have a type of Hölder inequality $\|u \cdot v\|_{X_c^1(a, b)} \leq \|u\|_{X_c^p(a, b)} \cdot \|v\|_{X_c^q(a, b)}$

Theorem 1.1. [10] Let $0 < a < b < +\infty$, $X_c^p(a, b) = L^p(a, b)$ and the norms $\|\cdot\|_{L^p[a, b]}$ and $\|\cdot\|_{X_c^p[a, b]}$ are equivalent.

Definition 1.3. [5] Let $0 < a < b < \infty, \alpha > 0$, the left and right Hadamard fractional integrals of order α of $u \in L^p(a, b)$ are respectively defined by

$${}^H I_{a^+}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s} \right)^{\alpha-1} \frac{u(s)}{s} ds \quad (a < t < b), \quad (1.2)$$

And

$${}^H I_{b^-}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\ln \frac{s}{t} \right)^{\alpha-1} \frac{u(s)}{s} ds \quad (a < t < b), \quad (1.3)$$

Definition 1.4. [5] Let $0 < a < b < \infty, \delta = t \frac{d}{dt}, n = [\alpha] + 1, [\alpha]$ denotes the integer part of the real number α . The left and right Hadamard fractional derivatives of order α of $u \in L^p(a, b)$ are respectively defined by:

$$\begin{aligned} {}^H D_{a^+}^\alpha u(t) &= \delta^n ({}^H I_{a^+}^{n-\alpha} u(t)) \\ &= \left(t \frac{d}{dt} \right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\ln \frac{t}{s} \right)^{n-\alpha-1} \frac{u(s)}{s} ds \quad a < t < b, \end{aligned} \quad (1.4)$$

And

$$\begin{aligned} {}^H D_{b^-}^\alpha u(t) &= (-\delta)^n ({}^H I_{b^-}^{n-\alpha} u(t)) \\ &= \left(-t \frac{d}{dt} \right)^n \frac{1}{\Gamma(n-\alpha)} \int_t^b \left(\ln \frac{s}{t} \right)^{n-\alpha-1} \frac{u(s)}{s} ds \quad a < t < b, \end{aligned} \quad (1.5)$$

property 1.1. If $\alpha, \beta > 0$ and $0 < a < b < +\infty$ then

$$\begin{aligned} \left({}^H I_{a^+}^\alpha \left(\ln \frac{x}{a} \right)^{\beta-1} \right) (t) &= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} \left(\ln \frac{t}{a} \right)^{\beta+\alpha-1}, \\ \left({}^H D_{a^+}^\alpha \left(\ln \frac{x}{a} \right)^{\beta-1} \right) (t) &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\ln \frac{t}{a} \right)^{\beta-\alpha-1}. \end{aligned}$$

And

$$\left({}^H I_{b^-}^\alpha \left(\ln \frac{b}{x} \right)^{\beta-1} \right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} \left(\ln \frac{b}{t} \right)^{\beta+\alpha-1},$$

$$\left({}^H D_{b^-}^\alpha \left(\ln \frac{b}{x} \right)^{\beta-1} \right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\ln \frac{b}{t} \right)^{\beta-\alpha-1}.$$

Remark 1.1. [5] If $\alpha > 0$ and $0 < a < b < +\infty$, when $0 < \alpha < 1$, on the other hand, for $j = [\alpha] + 1$, then

$$\left({}^H D_{a^+}^\alpha \left(\ln \frac{x}{a} \right)^{\alpha-j} \right)(t) = 0 \quad \text{and} \quad \left({}^H D_{b^-}^\alpha \left(\ln \frac{b}{x} \right)^{\alpha-j} \right)(t) = 0.$$

Corollary 1.1. [5] Let $\alpha > 0$, $n = [\alpha] + 1$ and $0 < a < b < +\infty$

The equality $\left({}^H D_{a^+}^\alpha u \right)(x) = 0$ is valid if, and only if, $u(x) = \sum_{j=1}^n c_j \left(\ln \frac{x}{a} \right)^{\alpha-j}$,

Where $c_j \in \mathbb{R}$, $(j = 1, \dots, n)$ are arbitrary constants.

The equality $\left({}^H D_{b^-}^\alpha u \right)(x) = 0$ is valid if, and only if, $u(x) = \sum_{j=1}^n d_j \left(\ln \frac{b}{x} \right)^{\alpha-j}$,

Where $d_j \in \mathbb{R}$, $(j = 1, \dots, n)$ are arbitrary constants.

In particular, when $0 < \alpha < 1$, the relation $\left({}^H D_{a^+}^\alpha u \right)(x) = 0$ holds if, and only if

$$u(x) = c \left(\ln \frac{x}{a} \right)^{\alpha-1}, \quad (c \in \mathbb{R})$$

The relation $\left({}^H D_{b^-}^\alpha u \right)(x) = 0$ holds if, and only if

$$u(x) = d \left(\ln \frac{b}{x} \right)^{\alpha-1}, \quad (d \in \mathbb{R})$$

Proposition 1.2. [10] For all $u \in L^p(a, b)$, we have ${}^H I_{a^+}^\alpha u, {}^H I_{b^-}^\alpha u \in L^p(a, b)$,

Moreover, if $\frac{1}{2} < \alpha < 1$ we have

$$\left\| {}^H I_{a^+}^\alpha u \right\|_{L^p(a,b)} \leq \frac{(b-a)^\alpha}{a^p b^{(\alpha-1)p} \Gamma(\alpha+1)} \|u\|_{L^p(a,b)}, \quad (1.6)$$

And

$$\left\| {}^H I_{b^-}^\alpha u \right\|_{L^p(a,b)} \leq \frac{(b-a)^\alpha}{a^p b^{(\alpha-1)p} \Gamma(\alpha+1)} \|u\|_{L^p(a,b)}. \quad (1.7)$$

Definition 1.5 . [5], [10] Let $f, g \in L^p(a, b)$, and $A, B \in \mathbb{R}$

$$AC(a, b) = \left\{ f : (a, b) \rightarrow \mathbb{R}, f(x) = c + \int_a^x \psi(x) dx, x \in (a, b), \right. \\ \left. c \in \mathbb{R}, \psi \in L^1(a, b) \right\},$$

$$AC_\delta(a, b) = \left\{ f : (a, b) \rightarrow \mathbb{R}, : \delta\psi \in AC(a, b) \right\}.$$

$${}^H AC_{a^+}^{\alpha, p}(a, b) = \left\{ u / u(x) = \frac{A}{\Gamma(\alpha)} \left(\ln \frac{x}{a} \right)^{\alpha-1} + \left({}^H I_{a^+}^\alpha f \right)(x) \right\},$$

$${}^H AC_{b^-}^{\alpha, p}(a, b) = \left\{ u / u(x) = \frac{B}{\Gamma(\alpha)} \left(\ln \frac{b}{x} \right)^{\alpha-1} + \left({}^H I_{b^-}^\alpha g \right)(x) \right\}.$$

property 1.2. [5] Let $\alpha > 0, \beta > 0$ and $0 < a < b < +\infty$. If $u \in L^p(a, b)$, then

$$\left({}^H D_{a^+}^\beta {}^H I_{a^+}^\alpha u \right)(x) = \left({}^H I_{a^+}^{\alpha-\beta} u \right)(x),$$

and

$$\left({}^H D_{b^-}^\beta {}^H I_{b^-}^\alpha u \right)(x) = \left({}^H I_{b^-}^{\alpha-\beta} u \right)(x).$$

In particular, if $\alpha = \beta$

$$\left({}^H D_{a^+}^\alpha {}^H I_{a^+}^\alpha u \right)(x) = u(x),$$

and

$$\left({}^H D_{b^-}^\alpha {}^H I_{b^-}^\alpha u \right)(x) = u(x).$$

Proposition 1.3 . [10] Let $p \geq 1, q \leq +\infty$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$. Then, for all

$u \in L^p(a, b), v \in L^q(a, b)$, we have:

$$\int_a^b u(t) \left({}^H I_{b^-}^\alpha v \right)(t) \frac{dt}{t} = \int_a^b v(t) \left({}^H I_{a^+}^\alpha u \right)(t) \frac{dt}{t} \quad (1.8)$$

Theorem 1.2 . [5] Let $\alpha > 0, n-1 \leq \alpha < n (n \in \mathbb{N}^*)$ and $0 < a < b < +\infty$.

$u \in L^p(a, b)$ and ${}^H I_{a^+}^{n-\alpha} u \in {}^H AC_\delta^n(a, b)$, then

$$\left({}^H I_{a^+}^\alpha {}^H D_{a^+}^\alpha u \right)(x) = u(x) - \sum_{k=1}^n \frac{\left(\delta^{n-k} \left({}^H I_{a^+}^{n-\alpha} u \right) \right)(a)}{\Gamma(\alpha - k + 1)} \left(\ln \frac{x}{a} \right)^{\alpha-k}. \quad (1.9)$$

In particular, if $\alpha = n \in \mathbb{N}$

$$\left({}^H I_{a^+}^n {}^H D_{a^+}^n u\right)(x) = u(x) - \sum_{k=1}^{n-1} \frac{(\delta^k u)(a)}{k!} \left(\ln \frac{x}{a}\right)^k. \quad (1.10)$$

If $\frac{1}{2} < \alpha < 1$, then

$$\left({}^H I_{a^+}^\alpha {}^H D_{a^+}^\alpha u\right)(x) = u(x) - \frac{({}^H I_{a^+}^{1-\alpha} u)(a)}{\Gamma(\alpha)} \left(\ln \frac{x}{a}\right)^{\alpha-1}. \quad (1.11)$$

Lemma 1.1. [10] Let $p \geq 1$, $q \leq +\infty$. be such that $\frac{1}{p} < \alpha$, then for all $u \in {}^H AC_{a^+}^{\alpha,p}(a,b)$ and $v \in L^2(a,b)$, we have

$$\begin{aligned} \int_a^b u(x) \left({}^H D_{b^-}^\alpha v(x)\right) \frac{dx}{x} &= \int_a^b \left({}^H D_{a^+}^\alpha u(x)\right) v(x) \frac{dx}{x} + \left({}^H I_{a^+}^{1-\alpha} u(a)\right) v(a) \\ &\quad - \left({}^H I_{b^-}^{1-\alpha} v(b)\right) u(b). \end{aligned} \quad (1.12)$$

Definition 1.6 . [9], [3] Let X be a Banach space. An operator $A: X \rightarrow X^*$ which satisfies

$$\langle Au - Av, u - v \rangle \geq 0 \quad (1.13)$$

For any $u, v \in X$ is called a monotone operator. An operator A is called strictly monotone if for $u \neq v$ strict inequality holds in (1.13). An operator A is called strongly monotone if there exists $c > 0$ such that

$$\langle Au - Av, u - v \rangle \geq c \|u - v\|^2$$

for any $u, v \in X$. It is clear that a strongly monotone is strictly monotone.

Definition 1.7. [9], [3] Let $A: X \rightarrow X^*$ be an operator on the real Banach space X .

a) A is said to be demicontinuous if

$$u_n \rightarrow u \text{ as } n \rightarrow +\infty \text{ implies } Au_n \rightarrow Au \text{ as } n \rightarrow +\infty.$$

b) A is said to be hemicontinuous if the real function

$$t \rightarrow \langle A(u + tv), w \rangle \text{ is continuous on } [0,1] \text{ for all } u, v, w \in X.$$

c) A is said to be coercive if

$$\lim_{\|u\| \rightarrow +\infty} \frac{\langle Au, u \rangle}{\|u\|} = +\infty$$

Remark 1.2. [4] , [3] It is easy to see that for monotone operator $A: X \rightarrow X^*$ with $Dom(A) = X$, demicontinuity and hemicontinuity are equivalent.

Theorem 1.3. [6] , [3] (Minty-Browder) Let be a reflexive real Banach space.

Let $A: X \rightarrow X^*$ be an operator which is bounded, hemicontinuous, coercive and monotone on the space X . Then, the equation $Au = f$ has at least one solution $u \in X$ for each $f \in X^*$. If A is strictly monotone then the solution is unique.

2. Variational structure:

Now we introduce a new space ${}^H E^\alpha$ which is suitable for the study of our fractional BVP.

Definition 2.1. Let $\frac{1}{2} < \alpha < 1$. The fractional derivative space ${}^H E^\alpha$ is defined by

$${}^H E^\alpha = \left\{ u \in L^2(a, b); {}^H D_{a^+}^\alpha u \in L^2(a, b); {}^H I_{a^+}^{1-\alpha} u(a) = u(b) = 0 \right\}. \quad (2.1)$$

with the norm

$$\|u\|_{{}^H E^\alpha} = \|u\|_\alpha = \left(\|u\|_{L^2}^2 + \|{}^H D_{a^+}^\alpha u\|_{L^2}^2 \right)^{\frac{1}{2}}. \quad (2.2)$$

Proposition 2.1. For all $u \in {}^H E^\alpha$, we have

$${}^H I_{a^+}^\alpha {}^H D_{a^+}^\alpha u = u \quad \text{and} \quad {}^H D_{a^+}^\alpha {}^H I_{a^+}^\alpha u = u.$$

Proof. The proof is easy by using (1.11) and (2.1)

Proposition 2.2. For all $u \in {}^H E^\alpha$, we have

$$\|u\|_{L^2} \leq \frac{(b-a)^\alpha}{a^2 b^{(\alpha-1)^2} \Gamma(\alpha+1)} \|{}^H D_{a^+}^\alpha u\|_{L^2}.$$

Proposition 2.3. There exist $\gamma_1, \gamma_2 \in \mathbb{R}^+$, such that

$$\gamma_1 \|{}^H D_{a^+}^\alpha u\|_{L^2} \leq \|u\|_\alpha \leq \gamma_2 \|{}^H D_{a^+}^\alpha u\|_{L^2}.$$

Proof. It's clear that $\|{}^H D_{a^+}^\alpha u\|_{L^2} \leq \|u\|_\alpha \Rightarrow \gamma_1 = 1$.

And we have $\|u\|_\alpha^2 = \|u\|_{L^2}^2 + \|{}^H D_{a^+}^\alpha u\|_{L^2}^2$, then

$$\begin{aligned} \exists c \in \mathbb{R}_+^* : \|u\|_\alpha &\leq c \left[\frac{(b-a)^\alpha}{a^2 b^{(\alpha-1)^2} \Gamma(\alpha+1)} + 1 \right] \|{}^H D_{a^+}^\alpha u\|_{L^2} \\ \Rightarrow \gamma_2 &= \left(\frac{(b-a)^\alpha}{a^2 b^{(\alpha-1)^2} \Gamma(\alpha+1)} + 1 \right). \end{aligned}$$

Remark 2.1. We can endowed ${}^H E^\alpha$ with respect to the equivalent norm $\|{}^H D_{a^+}^\alpha u\|_{L^2} = \|u\|_\alpha$.

Lemma 2.1. The space ${}^H E^\alpha$ is endowed with the structure of a Hilbert space together with the inner product

$$\begin{aligned} (\cdot, \cdot)_{H E^\alpha} : {}^H E^\alpha \times {}^H E^\alpha &\rightarrow \mathbb{R} \\ (u, v) &\rightarrow (u, v)_{H E^\alpha} = \int_a^b {}^H D_{a^+}^\alpha u(t) {}^H D_{a^+}^\alpha v(t) dt. \end{aligned}$$

Proposition 2.4. [10] If $u, v \in {}^H E^\alpha$, then

$$\int_a^b u(x) ({}^H D_{b^-}^\alpha v)(x) \frac{dx}{x} = \int_a^b ({}^H D_{a^+}^\alpha u)(x) v(x) \frac{dx}{x}. \quad (2.3)$$

Lemma 2.2. Let $\frac{1}{2} < \alpha < 1$ and $1 < p < \infty$. The fractional derivative space ${}^H E^\alpha$ is a reflexive and separable Banach space.

Lemma 2.3. For all $u \in {}^H E^\alpha$, we have that $({}^H E^\alpha[a, b], \|u\|_\alpha)$ embeds continuously in $(C[a, b], \|\cdot\|_\alpha)$.

Proof. For all $u \in {}^H E^\alpha$ we have

$$\begin{aligned} |{}^H I_{a^+}^\alpha {}^H D_{a^+}^\alpha u(t)| &= \left| \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s} \right)^{\alpha-1} \frac{{}^H D_{a^+}^\alpha u(s)}{s} ds \right| \\ &\leq \frac{1}{\Gamma(\alpha) \sqrt{s}} \left| \int_a^t \left(\ln \frac{t}{s} \right)^{\alpha-1} \frac{1}{\sqrt{s}} {}^H D_{a^+}^\alpha u(s) ds \right| \end{aligned}$$

Using Cauchy-Schwarz inequality

$$\begin{aligned} \left| {}^H I_{a^+}^\alpha {}^H D_{a^+}^\alpha u(t) \right| &\leq \frac{1}{\Gamma(\alpha)\sqrt{a}} \left(\int_a^b \left(\ln \frac{t}{s} \right)^{\alpha-1} \frac{1}{\sqrt{s}} ds \right)^{\frac{1}{2}} \left(\int_a^b \left| {}^H D_{a^+}^\alpha u(s) \right|^2 ds \right)^{\frac{1}{2}} \\ &\leq \frac{1}{\Gamma(\alpha)\sqrt{a}} \frac{1}{\sqrt{2\alpha-1}} \left(\ln \frac{b}{a} \right)^{\alpha-\frac{1}{2}} \|u\|_\alpha \end{aligned}$$

Hence

$$\|u\|_\infty \leq \frac{\left(\ln \frac{b}{a} \right)^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{a(2\alpha-1)}} \|u\|_\alpha$$

Lemma 2.4. For all $u \in {}^H E^\alpha$, we have that $({}^H E^\alpha[a, b], \|u\|_\alpha)$ compactly embedded in $(C[a, b], \|\cdot\|_\alpha)$.

Proof. From the preceding lemma, we obtain $v_n \rightarrow v$ in $(C[a, b], \|\cdot\|_\alpha)$ for given $v_n \rightarrow v$ in ${}^H E^\alpha$.

For $v_n \rightarrow v$ in ${}^H E^\alpha$ and $f \in (C[a, b], \|\cdot\|_\alpha)^*$, $f(v_n) \rightarrow f(v)$, which shows that $f \in ({}^H E^\alpha)^*$. Thus $(C[a, b], \|\cdot\|_\alpha)^* \subseteq ({}^H E^\alpha)^*$. Hence $v_n \rightharpoonup v$ in $(C[a, b], \|\cdot\|_\alpha)$ if $v_n \rightharpoonup v$ in ${}^H E^\alpha$. $\{v_n\}$ is bounded in $(C[a, b], \|\cdot\|_\alpha)$. This is because, using Banach-Steinhaus theorem, $\{v_n\}$ is bounded in ${}^H E^\alpha$.

Now to show that $\{v_n\}$ is uniformly equi-continuous, we have the following result.

Let $0 < a < t_1 < t_2 < b < \infty$, using Holder inequality for all $u \in (L^2[a, b], \|\cdot\|_\alpha)$, we have

$$\begin{aligned} \left| {}^H I_{a^+}^\alpha u(t_1) - {}^H I_{a^+}^\alpha u(t_2) \right| &= \frac{1}{\Gamma(\alpha)} \left| \int_a^{t_1} \left(\ln \frac{t_1}{x} \right)^{\alpha-1} u(x) \frac{dx}{x} - \int_a^{t_2} \left(\ln \frac{t_2}{x} \right)^{\alpha-1} u(x) \frac{dx}{x} \right| \\ &= \frac{1}{\Gamma(\alpha)} \left| \int_a^{t_1} \left(\ln \frac{t_1}{x} \right)^{\alpha-1} u(x) \frac{dx}{x} - \int_a^{t_1} \left(\ln \frac{t_2}{x} \right)^{\alpha-1} u(x) \frac{dx}{x} \right| + \frac{1}{\Gamma(\alpha)} \left| \int_{t_1}^{t_2} \left(\ln \frac{t_2}{x} \right)^{\alpha-1} u(x) \frac{dx}{x} \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \left| \left(\ln \frac{t_1}{x} \right)^{\alpha-1} - \left(\ln \frac{t_2}{x} \right)^{\alpha-1} \right| |u(x)| \frac{dx}{x} + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\ln \frac{t_2}{x} \right)^{\alpha-1} |u(x)| \frac{dx}{x} \end{aligned}$$

Using Cauchy-Schwartz inequality

$$\begin{aligned}
 & \left| {}^H I_{a^+}^\alpha u(t_1) - {}^H I_{a^+}^\alpha u(t_2) \right| \leq \\
 & \frac{\|u\|_{L^2}}{\Gamma(\alpha)\sqrt{a}} \left(\int_a^{t_1} \left(\left(\ln \frac{t_1}{x} \right)^{\alpha-1} - \left(\ln \frac{t_2}{x} \right)^{\alpha-1} \right)^2 \frac{1}{x} dx \right)^{\frac{1}{2}} + \frac{\|u\|_{L^2}}{\Gamma(\alpha)\sqrt{a}} \left(\int_{t_1}^{t_2} \left(\left(\ln \frac{t_2}{x} \right)^{\alpha-1} \right)^2 \frac{1}{x} dx \right)^{\frac{1}{2}} \\
 & \leq \frac{\|u\|_{L^2}}{\Gamma(\alpha)\sqrt{a}} \left(\int_a^{t_1} \left(\left(\ln \frac{t_1}{x} \right)^{2(\alpha-1)} - \left(\ln \frac{t_2}{x} \right)^{2(\alpha-1)} \right) \frac{1}{x} dx \right)^{\frac{1}{2}} + \frac{\|u\|_{L^2}}{\Gamma(\alpha)\sqrt{a}} \left(\int_{t_1}^{t_2} \left(\left(\ln \frac{t_2}{x} \right)^{2(\alpha-1)} \right) \frac{1}{x} dx \right)^{\frac{1}{2}} \\
 & \leq \frac{\|u\|_{L^2}}{\Gamma(\alpha)\sqrt{a(2\alpha-1)}} \left((\ln t_2) - (\ln t_1) \right)^{\alpha-\frac{1}{2}} + \frac{\|u\|_{L^2}}{\Gamma(\alpha)\sqrt{a(2\alpha-1)}} \left((\ln t_2) - (\ln t_1) \right)^{\alpha-\frac{1}{2}} \\
 & \leq 2 \frac{\|u\|_{L^2}}{\Gamma(\alpha)\sqrt{a(2\alpha-1)}} d^{\alpha-\frac{1}{2}} (t_2 - t_1)^{\alpha-\frac{1}{2}}
 \end{aligned}$$

where $d > 0$

Now we can prove that $\{u_n\}$ is uniformly equi-continuous. so, we have

$$\begin{aligned}
 |u(t_1) - u(t_2)| & \leq 2 \frac{\| {}^H D_{a^+}^\alpha u \|_{L^2}}{\Gamma(\alpha)\sqrt{a(2\alpha-1)}} d^{\alpha-\frac{1}{2}} (t_2 - t_1)^{\alpha-\frac{1}{2}} \\
 & = 2 \frac{\|u\|_\alpha}{\Gamma(\alpha)\sqrt{a(2\alpha-1)}} d^{\alpha-\frac{1}{2}} (t_2 - t_1)^{\alpha-\frac{1}{2}}.
 \end{aligned}$$

Then $|u(t_1) - u(t_2)| \leq c(t_2 - t_1)^{\alpha-\frac{1}{2}}$. where $d > 0$, so $\{u_n\}$ is relatively compact in $(C[a, b], \square)$ as a consequence of Ascoli-Arzelà theorem. Also every uniformly convergent sub sequence of $\{u_n\}$ converges uniformly to u on $[a, b]$. This completes the proof.

3. Main results:

Definition 3.1. A weak solution of the fractional boundary value problem (1.1) is given by a solution of the following variational formula

$$\int_a^b {}^H D_{a^+}^\alpha u(t) {}^H D_{a^+}^\alpha v(t) dt - \lambda \int_a^b f(t, u(t)) v(t) \frac{dt}{t} = 0, \quad \text{for all } v \in {}^H E^\alpha.$$

Theorem 3.1. Let's suppose that f satisfies hypotheses (H_1) and (H_2) . Then, problem (1.1) has a weak solution. With:

(H_1) There exist two functions $\gamma, \beta \in L^2[a, b]$ such that

$$|f(t, x)| \leq \gamma(t)|x| + \beta(t), \forall t \in [a, b], \forall x \in \mathbb{R}$$

With $\lambda < \frac{a\Gamma(\alpha)\sqrt{a(2\alpha-1)}}{\left(\ln \frac{b}{a}\right)^{\alpha-\frac{1}{2}} \|\gamma\|_{L^2}}.$

(H_2) $f : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ is decreasing with respect to the second variable, that is, $f(t, x_1) \leq f(t, x_2)$ for almost every $t \in [a, b]$ and $x_1, x_2 \in \mathbb{R}, x_1 \geq x_2$.

Proof. We define the operator $A: {}^H E^\alpha \rightarrow ({}^H E^\alpha)^*$ by $A = I - F$, where the operator I and F are defined from ${}^H E^\alpha$ into $({}^H E^\alpha)^*$ as

$$\langle I(u), v \rangle = \int_a^b {}^H D_{a^+}^\alpha u(t) {}^H D_{a^+}^\alpha v(t) dt$$

And

$$\langle F(u), v \rangle = \lambda \int_a^b f(t, u(t)) v(t) \frac{dt}{t}.$$

We search for a weak solution of problem (1.1) which is a function $u \in {}^H E^\alpha$ which satisfies the operator equation $Au = 0$.

We will use the Minty-Browder theorem and divide our proof into four steps.

Step 1. A is bounded.

Let $u \in {}^H E^\alpha$, be such that $\|u\|_\alpha < R, (R > 0)$. Using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
 \|I(u)\|_{(H E^\alpha)^*} &= \sup_{\|v\|_\alpha \leq 1} |\langle I(u), v \rangle| \\
 &= \sup_{\|v\|_\alpha \leq 1} \left| \int_a^b {}^H D_{a^+}^\alpha u(t) {}^H D_{a^+}^\alpha v(t) dt \right| \\
 &\leq \sup_{\|v\|_\alpha \leq 1} \| {}^H D_{a^+}^\alpha u \|_{L^2} \| {}^H D_{a^+}^\alpha v \|_{L^2} \\
 &\leq \|u\|_\alpha \leq R.
 \end{aligned}$$

And

$$\begin{aligned}
 \|F(u)\|_{(H E^\alpha)^*} &= \sup_{\|v\|_\alpha \leq 1} |\langle F(u), v \rangle| = \sup_{\|v\|_\alpha \leq 1} \left| \lambda \int_a^b f(t, u(t)) v(t) \frac{dt}{t} \right| \\
 &\leq \sup_{\|v\|_\alpha \leq 1} \frac{\lambda}{a} \left(\int_a^b |\gamma(t)| |u(t)| |v(t)| dt + \int_a^b |\beta(t)| |v(t)| dt \right) \\
 &\leq \sup_{\|v\|_\alpha \leq 1} \frac{\lambda}{a} \left[\|u\|_\infty \int_a^b |\gamma(t)| |v(t)| dt + \int_a^b |\beta(t)| |v(t)| dt \right] \\
 &\leq \sup_{\|v\|_\alpha \leq 1} \|v\|_\alpha \frac{\lambda}{a} \left[\frac{\left(\ln \frac{b}{a} \right)^{\alpha-\frac{1}{2}}}{\Gamma(\alpha) \sqrt{a(2\alpha-1)}} \|u\|_\alpha \left(\int_a^b |\gamma(t)|^2 dt \right)^{\frac{1}{2}} + \left(\int_a^b |\beta(t)|^2 dt \right)^{\frac{1}{2}} \right] \\
 &\leq \frac{\lambda \left(\ln \frac{b}{a} \right)^{\alpha-\frac{1}{2}}}{a \Gamma(\alpha) \sqrt{a(2\alpha-1)}} \|u\|_\alpha \|\gamma\|_{L^2} + \frac{\lambda}{a} \|\beta\|_{L^2}.
 \end{aligned}$$

Hence A is bounded.

Step 2. A is demicontinuous.

We prove that F is strongly continuous that is, if $u_n \rightarrow n$, then $F(u_n) \rightarrow F(n)$. Let (u_n) be such that $u_n \rightarrow n$ in ${}^H E^\alpha$. Now (u_n) is bounded in ${}^H E^\alpha$ and by Lemma 2.3, we have that (u_n) is bounded in $C([a, b], \mathbb{R})$, and by Lemma 2.4, we have $u_n \rightarrow n$ in $C([a, b], \mathbb{R})$. Then

$$\|F(u) - F(u_n)\|_{(H E^\alpha)^*} = \sup_{\|v\|_\alpha \leq 1} |\langle F(u) - F(u_n), v \rangle|$$

$$\begin{aligned}
 &= \sup_{\|v\|_\alpha \leq 1} \left| \lambda \int_a^b (f(t, u(t)) - f(t, u_n(t))) v(t) \frac{dt}{t} \right| \\
 &\leq \frac{\lambda}{a} \sup_{\|v\|_\alpha \leq 1} \int_a^b |\gamma(t)| |u(t)| |v(t)| dt + \frac{\lambda}{a} \sup_{\|v\|_\alpha \leq 1} \int_a^b |\gamma(t)| |u_n(t)| |v(t)| dt \\
 &\quad + \frac{2\lambda}{a} \sup_{\|v\|_\alpha \leq 1} \int_a^b |\beta(t)| |v(t)| dt \\
 &\leq \frac{\lambda \left(\ln \frac{b}{a} \right)^{\alpha-\frac{1}{2}}}{a \Gamma(\alpha) \sqrt{a(2\alpha-1)}} \|\gamma\|_{L^2} C + \frac{2\lambda}{a} \|\beta\|_{L^2}
 \end{aligned}$$

for some constant $C > 0$. Since $u_n \rightarrow u$ in $C([a, b], \mathbb{R})$, we obtain

$$\lambda \int_a^b (f(t, u(t)) - f(t, u_n(t))) v(t) \frac{dt}{t} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus F is strongly continuous and therefore it is continuous. From the fact that I is continuous, we deduce that the operator A is continuous. Thus A is demicontinuous.

Step 3. A is monotone.

$$\text{Note } \langle I(u) - I(v), u - v \rangle = \int_a^b ({}^H D_{a^+}^\alpha u(t) - {}^H D_{a^+}^\alpha v(t))^2 dt = \|u - v\|_\alpha^2,$$

so, I is strongly monotone.

Also, since f is decreasing with respect to the second variable,

$$\langle F(u) - F(v), u - v \rangle = \lambda \int_a^b (f(t, u(t)) - f(t, v(t))) (u(t) - v(t)) \frac{dt}{t} \leq 0.$$

Thus A is strongly monotone.

Step 4. A is a coercive operator.
 Now

$$\frac{1}{\|u\|_\alpha} \langle A(u), u \rangle = \frac{1}{\|u\|_\alpha} \left[\int_a^b |{}^H D_{a^+}^\alpha u(t)|^2 dt - \lambda \int_a^b f(t, u(t)) u(t) \frac{dt}{t} \right]$$

$$\begin{aligned}
 &\geq \frac{1}{\|u\|_\alpha} \left[\|u\|_\alpha^2 - \frac{\lambda}{a} \int_a^b (|\gamma(t)| |u(t)| |u(t)| + |\beta(t)| |u(t)|) dt \right] \\
 &\geq \frac{1}{\|u\|_\alpha} \left[\|u\|_\alpha^2 - \frac{\lambda}{a} \|\gamma\|_{L^2} \|u\|_\infty \|u\|_{L^2} - \frac{\lambda}{a} \|\beta\|_{L^2} \|u\|_{L^2} \right] \\
 &\geq \frac{1}{\|u\|_\alpha} \left[\|u\|_\alpha^2 - \frac{\lambda \left(\ln \frac{b}{a} \right)^{\alpha-\frac{1}{2}}}{a \Gamma(\alpha) \sqrt{a(2\alpha-1)}} \|\gamma\|_{L^2} \|u\|_\alpha^2 - \frac{\lambda}{a} \|\beta\|_{L^2} \|u\|_\alpha \right] \\
 &\quad \left(1 - \frac{\lambda \left(\ln \frac{b}{a} \right)^{\alpha-\frac{1}{2}}}{a \Gamma(\alpha) \sqrt{a(2\alpha-1)}} \|\gamma\|_{L^2} \right) \|u\|_\alpha - \frac{\lambda}{a} \|\beta\|_{L^2}
 \end{aligned}$$

so A is coercive.

The assumptions in the statement of Theorem 3.1 are fulfilled, so problem (1.1) has a weak solution. Moreover, A since is strictly monotone, the solution is unique.

4. Example:

$$\begin{cases} {}^H D_{e^-}^{\frac{4}{5}} \left(t {}^H D_{1^+}^{\frac{4}{5}} u(t) \right) = \frac{1}{10} (-u(t) |u(t)|), & t \in [1, e], \\ {}^H I_{1^+}^{\frac{4}{5}} u(1) = u(e) = 0, \end{cases} \quad (4.1)$$

Here $\alpha = \frac{4}{5}$, $[a, b] = [1, e]$, $\lambda = \frac{1}{10}$, and $f(t, u) = -u|u|$

$$\gamma(t) = 2t, \quad \beta(t) = 3, \quad \lambda = \frac{1}{10} < \frac{a \Gamma(\alpha) \sqrt{a(2\alpha-1)}}{\left(\ln \frac{b}{a} \right)^{\alpha-\frac{1}{2}} \|\gamma\|_{L^2}} \approx 0,179.$$

So all the requirements of Theorem 3.1 are satisfied, Hence problem (4.1) has a weak solution as a consequence of Theorem 3.1.

Conclusion:

In this paper, we have investigated a boundary value problem involving the Hadamard fractional derivative defined on a bounded interval. By employing the Minty–Browder theorem, we established rigorous results concerning the existence and uniqueness of weak solutions. The main contribution of this work lies in extending the applicability of monotone operator theory to fractional differential equations with Hadamard-type derivatives, which are less explored compared to the Riemann–Liouville and Caputo frameworks.

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