



**Portfolio Optimization And International Diversification:
A Markowitz-Based Analysis.**

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Abstract:

This article explores the effectiveness of international diversification in portfolio management based on Markowitz's theory. By analyzing the correlation between 13 global financial indices, the study identifies a strong integration among 9 of them, limiting the benefits of diversification. As a result, only 5 indices were selected for portfolio construction. Several portfolios were then developed: without constraints, with a risk-free asset, and under inequality constraints. The results show that international diversification reduces risk by at least 30%, while the inclusion of constraints restricts the benefits of diversification without eliminating them. The study also highlights the critical role of market correlation in optimizing portfolio performance.

Keywords: Portfolio optimization, diversification, Markowitz theory, correlation, risk-free asset, efficient frontier.

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1. Introduction:

The management of financial portfolios involves allocating capital in financial markets where investors seeking returns meet agents offering savings. The key challenge lies in selecting the optimal investment and diversification strategies and determining the ideal portfolio composition through the appropriate weighting of financial assets. Making such decisions requires a thorough understanding of the various financial instruments, their characteristics, and the dynamics of the markets in which they operate.

Portfolio theory, particularly the Mean-Variance framework introduced by Markowitz, formalizes the relationship between an asset's attractiveness—measured by expected return—and its risk—quantified by variance or standard deviation. This theory guides rational investors in achieving effective diversification to optimize their investment outcomes. However, since securities within a single domestic market often exhibit strong positive correlations, the diversification benefits remain limited. This phenomenon reduces the efficacy of traditional portfolio optimization within a single market and motivates investors to seek international diversification, targeting markets that display weaker or even negative correlations.

1.1. Statement of the Problem:

In an increasingly globalized financial environment, investors face the critical challenge of managing portfolio risk while striving to maximize returns. Traditional portfolio theory demonstrates that diversification can reduce risk; however, domestic markets often exhibit high correlations among assets, limiting the benefits of diversification within a single country. This raises the question of whether and to what extent international diversification across various global financial markets can effectively improve portfolio performance. Furthermore, understanding how constraints such as investment limits and the presence of risk-free assets influence the construction and efficiency of these portfolios remains an open challenge. This study addresses the gap by investigating the role of international diversification in optimizing portfolio composition and risk-return trade-offs under realistic investment constraints.

1.2. Research Questions:

What is the contribution of international diversification to the composition and performance of investment portfolios? How does diversification across different global financial markets impact portfolio risk and return?

1.3. Hypotheses:

- International diversification improves portfolio efficiency by reducing overall risk for a given level of expected return.
- The correlations among major international financial markets vary sufficiently to provide meaningful diversification benefits.

1.4. Significance of the Study:

This study contributes to the understanding of international portfolio diversification by empirically examining the impact of including multiple global financial indices in portfolio construction. The findings will inform fund managers and investors about the potential risk-reduction and return-enhancement benefits achievable through global asset allocation, especially in an increasingly integrated world economy.

1.5. Objectives of the Study:

- To analyze the correlation structure among selected major international financial indices.
- To construct and compare efficient portfolios based on international diversification strategies, with and without the presence of a risk-free asset.
- To assess the impact of portfolio constraints on diversification outcomes.

1.6. Methodology:

The study employs empirical portfolio optimization using historical data from thirteen major international indices, subsequently narrowed down to five representative indices based on their correlation characteristics. The Mean-Variance optimization model is applied, incorporating both unconstrained and constrained portfolio frameworks. Risk and return are evaluated in continuous time using logarithmic returns, with portfolio performance assessed by variance and expected returns.

1.7. Literature Review:

Previous research has highlighted the limits of domestic diversification due to market integration and correlation structures (Markowitz, 1952; Solnik, 1974). Studies emphasize the potential for international diversification to improve portfolio performance, although benefits are contingent on the dynamic interdependence of global markets (Harvey, 1995; Bekaert & Harvey, 1997). This study builds on this foundation by providing a contemporary analysis of portfolio construction using a diversified set of international indices.

1.8. Organization of the Study:

The paper is structured as follows: Section 2 details the data and methodology used for portfolio construction. Section 3 presents empirical results on portfolio efficiency under different constraints and scenarios. Section 4 discusses the implications of these findings for investors. The final section concludes with recommendations and avenues for future research.

2. Investment Portfolio Composition:

From a set of securities, it is possible to create an infinite number of portfolios. The criterion of efficiency is to identify among them those that are of interest to the investor. For an investor who values expected return and is averse to risk, a portfolio is efficient when its expected return is maximized for a given level of risk.

2.1. Data Presentation:

From the diverse range of international financial markets, we selected 13 of the most important financial markets (represented by stock indices) that, in our view, meet the key criteria for international diversification, namely ensuring geographic diversification, currency diversification, and diversification among financial markets of countries at different levels of development. The corresponding financial indices are: *Dow Jones Industrial Average*, *Nasdaq Composite*, *S&P 500*, *Russell*, *Euro Stoxx 50*, *FTSE 100*, *DAX*, *CAC 40*, *Nikkei 225*, *Shanghai Composite*, *Nifty 50*, *Tadawul All Share*, and *Bovespa*.

The analysis of historical correlation coefficients among these 13 financial indices shows that 9 indices (*Dow Jones Industrial Average*, *Nasdaq Composite*, *S&P 500*, *Russell 2000*, *DAX*, *CAC 40*, *Nikkei 225*, *Nifty 50*, and *Bovespa*) have reached a very high level of integration. This integration is reflected by positive correlation coefficients close to 1 (> 0.8)¹. According to Markowitz's portfolio management theory, diversification among these 9 indices does not significantly improve the expected portfolio return nor reduce the corresponding risk. Therefore, it is reasonable to invest in only one index among them. To increase diversification efficiency, we chose the *Nasdaq Composite* index to represent these 9 indices since it is on average the least correlated with the other 4 indices. In the end, we retained 5 indices for the construction of our portfolios: *Nasdaq Composite*, *Euro Stoxx 50*, *FTSE 100*, *Shanghai Composite*, and *Tadawul All Share*.

The returns of the 5 selected indices are calculated continuously according to the formula:

$$R_t = \ln\left(\frac{C_t}{C_{t-1}}\right) \dots \dots \dots (1)$$

where $\ln$ is the natural logarithm.

¹ We arbitrarily set this correlation threshold.

The following table shows the expected returns for the 5 corresponding financial indices².

Table N°01: Returns for the Year 2024

Indices	Returns R_T
<i>R_NASDAQ_COMPOSITE</i>	14,06%
<i>R_EURO_STOXX_50</i>	6,91%
<i>R_FTSE_100</i>	4,22%
<i>R_SHANGHAI_COMPOSITE</i>	5,07%
<i>R_TADAWUL</i>	10,23%

Source: Calculated based on data extracted from the website
<https://fr.investing.com> (consulted in 05/01/2025)

In Markowitz theory, the values of the variance-covariance matrix and the values of the correlation matrix of asset returns play an important role in portfolio management and determine the effectiveness of diversification. The following two tables summarize these values respectively for the 5 return series³.

Table N°02: Historical Correlation Matrix between the 5 Return Series

	<i>R_NASDAQ_COMPOSITE</i>	<i>R_EURO_STOXX_50</i>	<i>R_FTSE_100</i>	<i>R_SHANGHAI_COMPOSITE</i>	<i>R_TADAWUL</i>
<i>R_NASDAQ_COMPOSITE</i>	1,000	0,583	0,506	0,241	0,290
<i>R_EURO_STOXX_50</i>		1,000	0,866	0,259	0,432
<i>R_FTSE_100</i>			1,000	0,267	0,426
<i>R_SHANGHAI_COMPOSITE</i>				1,000	0,222
<i>R_TADAWUL</i>					1,000

Source: Calculated based on data extracted from the website
<https://fr.investing.com> (consulted in 05/01/2025)

² In practice, one must forecast future returns, which remains a delicate task given the efficiency of financial markets.

³ The choice between financial indices was made based on the correlation between their values, not between their return values.

Table N°03: Variance-Covariance Matrix of the 5 Return Series

	<i>R_NASDAQ_COMPOSITE</i>	<i>R_EURO_STOXX_50</i>	<i>R_FTSE_100</i>	<i>R_SHANGHAI_COMPOSITE</i>	<i>R_TADAWUL</i>
<i>R_NASDAQ_COMPOSITE</i>	0,0002624	0,0001048	0,0000590	0,0000590	0,0000630
<i>R_EURO_STOXX_50</i>		0,0001232	0,0000692	0,0000434	0,0000643
<i>R_FTSE_100</i>			0,0000518	0,0000290	0,0000411
<i>R_SHANGHAI_COMPOSITE</i>				0,0002280	0,0000449
<i>R_TADAWUL</i>					0,0001796

Source: Calculated based on data extracted from the website
<https://fr.investing.com> (consulted 05/01/2025)

2.2. Efficient Portfolios in the Absence of Inequality Constraints:

Portfolio composition in the absence of inequality constraints is distinguished by two cases: portfolio composition without and with the presence of a risk-free asset.

2.2.1. Portfolio in the Absence of a Risk-Free Asset:

Let us denote by X_1, X_2, X_3, X_4 and X_5 the proportions to be invested in the financial indices *Nasdaq Composite*, *Euro Stoxx 50*, *FTSE 100*, *Shanghai Composite*, and *Tadawul All Share*, respectively. Let $\sigma_{R_{pf}}^2$ and E_{pf} denote the variance (risk) and the expected return of the composed portfolio. The objective is to minimize the variance of the portfolio return, expressed as:

$$\begin{aligned} \sigma_{R_{pf}}^2 = & (0,0162)^2 * X_1^2 + (0,0111)^2 * X_2^2 + (0,0072)^2 * X_3^2 + (0,0151)^2 * X_4^2 + (0,0134)^2 * X_5^2 \\ & + 2X_1X_2(0,0001048) + 2X_1X_3(0,0000590) + 2X_1X_4(0,0000590) \\ & + 2X_1X_5(0,0000630) + 2X_2X_3(0,0000692) + 2X_2X_4(0,0000434) \\ & + 2X_2X_5(0,0000643) + 2X_3X_4(0,0000290) + 2X_3X_5(0,0000411) \\ & + 2X_4X_5(0,0000449) \end{aligned}$$

Under the two constraints:

- $E_{pf} = 0,1406X_1 + 0,0691X_2 + 0,0422X_3 + 0,0507X_4 + 0,1023X_5$, which reaches the level E^* ;
- $\sum_{i=1}^5 X_i = 1$

The objective function is therefore as follows:

$$Z = \sigma_{R_{pf}}^2 + \lambda_1(E_{pf} - E^*) + \lambda_2\left(\sum_{i=1}^5 X_i - 1\right)$$

And the elements of the system are:

$$C = \begin{pmatrix} 0,0005249 & 0,0002097 & 0,0001180 & 0,0001179 & 0,0001259 & 0,1406 & 1 \\ 0,0002097 & 0,0002464 & 0,0001384 & 0,0000868 & 0,0001285 & 0,0691 & 1 \\ 0,0001180 & 0,0001384 & 0,0001037 & 0,0000581 & 0,0000822 & 0,0422 & 1 \\ 0,0001179 & 0,0000868 & 0,0000581 & 0,0004560 & 0,0000898 & 0,0507 & 1 \\ 0,0001259 & 0,0001285 & 0,0000822 & 0,0000898 & 0,0003591 & 0,1023 & 1 \\ 0,1406 & 0,0691 & 0,0422 & 0,0507 & 0,1023 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix};$$

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} \text{ and } K = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ E^* \\ 1 \end{pmatrix}$$

To obtain the solution $(X_{i,\overline{i=1,5}})$ corresponding to an efficient portfolio, it is sufficient to assign a value to E^* and multiply the inverse matrix C^{-1} by the vector K . The following table describes the structure of efficient portfolios corresponding to different levels of expected return E_{pf}^* and the corresponding risk values (standard deviation σ_{pf}).

Table N°04: Efficient Portfolios Without Inequality Constraints and Without a Risk-Free Asset

E_{pf}^*	X_1	X_2	X_3	X_4	X_5	σ_{pf}
1%	-0,13	-0,62	1,70	0,10	-0,05	0,71%
2%	-0,08	-0,57	1,55	0,09	-0,00	0,67%
<u>4%</u>	<u>0,04</u>	<u>-0,48</u>	<u>1,26</u>	<u>0,08</u>	<u>0,09</u>	<u>0,64%⁴</u>
6%	-0,01	-0,51	1,38	0,09	0,05	0,64%
8%	0,28	-0,29	0,66	0,06	0,29	0,82%
10%	0,40	-0,19	0,36	0,05	0,39	0,99%
12%	0,52	-0,10	0,06	0,03	0,49	1,17%
14%	0,64	-0,01	-0,24	0,02	0,58	1,38%
16%	0,76	0,09	-0,54	0,01	0,68	1,59%

Source: Calculated by the author

⁴ Minimum-Risk Portfolio Measured by Standard Deviation.

The set of pairs (E_{pf}^*, σ_{pf}) constitutes the set of efficient portfolios, or the efficient frontier. Each fund manager is required to choose a single portfolio on the efficient frontier based on their attitude toward risk, represented by their indifference curve. By comparing the values from Tables N°01 and N°04, we can observe that international diversification reduces investment risk by an average of 22% compared to investing in a single financial market.

2.2.2. Portfolios in the Presence of a Risk-Free Asset:

A risk-free asset is an asset whose return is known in advance and for which the counterparty default risk is considered to be zero. Governments or certain fund managers are often required to hold a portion of risk-free assets in their portfolios; this proportion is usually set by law to limit the fund's exposure to risk. Government bonds are generally considered an example of a risk-free asset.

In our case, we take the most realistic situation for the fund manager where risk-free assets exist. We consider the U.S. Treasury Bill (T-Bill) as a risk-free asset. It is a short-term debt security, with a maturity of one year or less, issued by the U.S. Government. The T-Bill is recognized by investors as the least risky Treasury obligation in the United States.

We denote by X_6 the position taken by the investor in the risk-free asset T-Bill, and by R_f its rate of return, with $R_f = 5,02\%$ corresponding to the daily quoted rate on the secondary market for the 26-week maturity as of 02/16/2025. The objective function is therefore expressed as:

$$Z = \sigma_{R_{pf}}^2 + \lambda_1(E_{pf} - E^*) + \lambda_2 \left(\sum_{i=1}^6 X_i - 1 \right) \dots \dots (2)$$

With

$$\begin{aligned} \sigma_{R_{pf}}^2 = & (0,0162)^2 * X_1^2 + (0,0111)^2 * X_2^2 + (0,0072)^2 * X_3^2 + (0,0151)^2 * X_4^2 + (0,0134)^2 * X_5^2 \\ & + 2X_1X_2(0,0001048) + 2X_1X_3(0,0000590) + 2X_1X_4(0,0000590) \\ & + 2X_1X_5(0,0000630) + 2X_2X_3(0,0000692) + 2X_2X_4(0,0000434) \\ & + 2X_2X_5(0,0000643) + 2X_3X_4(0,0000290) + 2X_3X_5(0,0000411) \\ & + 2X_4X_5(0,0000449) \end{aligned}$$

Under the two constraints:

- $E_{pf} = 0,1406X_1 + 0,0691X_2 + 0,0422X_3 + 0,0507X_4 + 0,1023X_5 + 0,0502X_6$, which reaches the level E^* ;
- $\sum_{i=1}^6 X_i = 1$.

Knowing that:

$$= \begin{pmatrix} 0,0005249 & 0,0002097 & 0,0001180 & 0,0001179 & 0,0001259 & 0 & 0,1406 & 1 \\ 0,0002097 & 0,0002464 & 0,0001384 & 0,0000868 & 0,0001285 & 0 & 0,0691 & 1 \\ 0,0001180 & 0,0001384 & 0,0001037 & 0,0000581 & 0,0000822 & 0 & 0,0422 & 1 \\ 0,0001179 & 0,0000868 & 0,0000581 & 0,0004560 & 0,0000898 & 0 & 0,0507 & 1 \\ 0,0001259 & 0,0001285 & 0,0000822 & 0,0000898 & 0,0003591 & 0 & 0,1023 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0,0502 & 1 \\ 0,1406 & 0,0691 & 0,0422 & 0,0507 & 0,1023 & 0,0502 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} \text{ and } K = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ E^* \\ 1 \end{pmatrix}$$

The solution of the system for different levels of expected return and the corresponding risks is summarized in the following table:

Table N°05: Efficient Portfolios Without Constraints and With a Risk-Free Asset.

E_{pf}^*	X_1	X_2	X_3	X_4	X_5	X_6	σ_{pf}
5,02%	0,00	0,00	0,00	0,00	0,00	1,00	0,00%
6%	0,05	0,06	-0,19	-0,01	0,04	1,04	0,11%
8%	0,16	0,20	-0,59	-0,03	0,13	1,13	0,34%
10%	0,27	0,33	-0,99	-0,05	0,21	1,22	0,57%
12%	0,38	0,46	-1,39	-0,07	0,30	1,31	0,79%
14%	0,49	0,59	-1,79	-0,09	0,38	1,40	1,02%
16%	0,60	0,73	-2,18	-0,10	0,47	1,49	1,25%
18%	0,71	0,86	-2,58	-0,12	0,55	1,58	1,48%
20%	0,82	0,99	-2,98	-0,14	0,64	1,67	1,70%

Source: Calculated by the author

By comparing the results of Tables N°04 and N°05, we observe that the introduction of the risk-free asset has modified the efficient frontier of the investor's investment opportunities, while gaining approximately 2% additional return for the same level of risk. Also, for values of $(0 \leq X_{i;\overline{i=1,5}} \leq 1)$, the fund investor will lend the corresponding portion of their holdings at the risk-free rate, and a value $(X_{i;\overline{i=1,5}} < 0)$ means that the investor borrows at the rate $R_f = 5,02\%$, allowing them to invest more than their initial holdings.

2.3. Efficient Portfolios in the Presence of Inequality Constraints:

For an investor, it often happens that the invested proportions X_i are subject to inequality constraints. In fact, to prevent overly speculative positions, X_i are prohibited from taking negative values or exceeding fixed percentages for various reasons.

In this case, the objective function of the optimization problem becomes:

$$\begin{aligned} Z = & -\lambda(0,1406X_1 + 0,0691X_2 + 0,0422X_3 + 0,0507X_4 + 0,1023X_5) + (0,0162)^2 * X_1^2 \\ & + (0,0111)^2 * X_2^2 + (0,0072)^2 * X_3^2 + (0,0151)^2 * X_4^2 + (0,0134)^2 * X_5^2 \\ & + 2X_1X_2(0,0001048) + 2X_1X_3(0,0000590) + 2X_1X_4(0,0000590) \\ & + 2X_1X_5(0,0000630) + 2X_2X_3(0,0000692) + 2X_2X_4(0,0000434) \\ & + 2X_2X_5(0,0000643) + 2X_3X_4(0,0000290) + 2X_3X_5(0,0000411) \\ & + 2X_4X_5(0,0000449) + \gamma\left(\sum_{i=1}^5 X_i - 1\right) \end{aligned}$$

The derivatives of the function Z with respect to $X_{i;\overline{i=1,5}}$ $\left(\frac{\delta Z}{\delta X_{i;\overline{i=1,5}}}\right)$ and with respect to γ

$\left(\frac{\delta Z}{\delta \gamma}\right)$ are given by:

$$\begin{cases} \frac{\delta Z}{\delta X_1} = -0,1406\lambda + 0,0005249X_1 + 0,0002097X_2 + 0,0001180X_3 + 0,0001179X_4 + 0,0001259X_5 + \gamma \\ \frac{\delta Z}{\delta X_2} = -0,0691\lambda + 0,0002097X_1 + 0,0002464X_2 + 0,0001384X_3 + 0,0000868X_4 + 0,0001285X_5 + \gamma \\ \frac{\delta Z}{\delta X_3} = -0,0422\lambda + 0,0001180X_1 + 0,0001384X_2 + 0,0001037X_3 + 0,0000581X_4 + 0,0000822X_5 + \gamma \\ \frac{\delta Z}{\delta X_4} = -0,0507\lambda + 0,0001179X_1 + 0,0000868X_2 + 0,0000581X_3 + 0,0004560X_4 + 0,0000898X_5 + \gamma \\ \frac{\delta Z}{\delta X_5} = -0,1023\lambda + 0,0001259X_1 + 0,0001285X_2 + 0,0000822X_3 + 0,0000898X_4 + 0,0003591X_5 + \gamma \\ \frac{\delta Z}{\delta \gamma} = X_1 + X_2 + X_3 + X_4 + X_5 - 1 \end{cases}$$

Following the steps of the Markowitz algorithm, we assign in the first step an 'in' status to the NASDAQ_COMPOSITE index, since it has the highest expected return of 14.06% (see Table N°04 above), and a 'down' status to the other four indices. The starting solution of the system is:

$$C = \begin{pmatrix} 0,0005249 & 0,0002097 & 0,0001180 & 0,0001179 & 0,0001259 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}; X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ \gamma \end{pmatrix};$$

$$K = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } E = \begin{pmatrix} 0,1406 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The first solution is given by:

$$\begin{cases} X_1 = 1 \\ X_2 = 0 \\ X_3 = 0 \\ X_4 = 0 \\ X_5 = 0 \\ \gamma = 0,1406\lambda - 0,0005249 \end{cases}$$

The critical λ values for the 5 considered indices are calculated by substituting the elements of the first solution into the equation $\left(\frac{\delta Z}{\delta X_{i,i=1,5}} = 0\right)$. We obtain:

$$\begin{cases} \frac{\delta Z}{\delta X_1} = -0,1406\lambda + 0,0005249 + 0,1406\lambda - 0,0005249 = 0 \\ \frac{\delta Z}{\delta X_2} = -0,0691\lambda + 0,0002097 + 0,1406\lambda - 0,0005249 = 0 \\ \frac{\delta Z}{\delta X_3} = -0,0422\lambda + 0,0001180 + 0,1406\lambda - 0,0005249 = 0 \\ \frac{\delta Z}{\delta X_4} = -0,0507\lambda + 0,0001179 + 0,1406\lambda - 0,0005249 = 0 \\ \frac{\delta Z}{\delta X_5} = -0,1023\lambda + 0,0001259 + 0,1406\lambda - 0,0005249 = 0 \\ \frac{\delta Z}{\delta \gamma} = X_1 + X_2 + X_3 + X_4 + X_5 - 1 \end{cases}$$

$$\begin{cases} \frac{\delta Z}{\delta X_2} = 0 \Rightarrow \lambda_{c,2} = 0,0044 \\ \frac{\delta Z}{\delta X_3} = 0 \Rightarrow \lambda_{c,3} = 0,0041 \\ \frac{\delta Z}{\delta X_4} = 0 \Rightarrow \lambda_{c,4} = 0,0045 \\ \frac{\delta Z}{\delta X_5} = 0 \Rightarrow \lambda_{c,5} = 0,0104 \end{cases}$$

For X_1 , there is no critical λ for this asset. The highest critical λ is equal to 0.0104, so the first solution is valid in the interval $[0,0104, +\infty[$. At this value of λ , the corresponding corner portfolio has performance identical to that of the *NASDAQ_COMPOSITE* index, since it is the only index included in the investment portfolio.

The first solution ceases to be valid when λ reaches 0.0104. At this value, the first corner portfolio (1) is obtained. The structure of this portfolio is:

$$\begin{aligned} X_1 &= 1 \\ X_2 &= 0 \\ X_3 &= 0 \\ X_4 &= 0 \\ X_5 &= 0 \end{aligned}$$

Starting from $\lambda = 0,0104$, the variable X_5 , corresponding to the *Tadawul All Share* index, acquires an 'in' status, while the status of the other variables remains unchanged. The matrix C and the vectors K and E become:

$$C = \begin{pmatrix} 0,0005249 & 0,0002097 & 0,0001180 & 0,0001179 & 0,0001259 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0,0001259 & 0,0001285 & 0,0000822 & 0,0000898 & 0,0003591 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix};$$

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ \gamma \end{pmatrix}; K = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ et } E = \begin{pmatrix} 0,1406 \\ 0 \\ 0 \\ 0 \\ 0,1023 \\ 0 \end{pmatrix}$$

Thus, the second solution is given by:

$$\begin{cases} X_1 = 60,582\lambda + 0,369 \\ X_2 = 0 \\ X_3 = 0 \\ X_4 = 0 \\ X_5 = -60,582\lambda + 0,631 \\ \gamma = 0,116\lambda - 0,00027 \end{cases}$$

We once again calculate the critical λ values for the different assets:

$$\frac{\delta Z}{\delta X_1} = -0,1406\lambda + 0,0005249(60,582\lambda + 0,369) + 0,0001259(-60,582\lambda + 0,631) + 0,116\lambda - 0,00027$$

$$\frac{\delta Z}{\delta X_2} = -0,0691\lambda + 0,0002097(60,582\lambda + 0,369) + 0,0001285(-60,582\lambda + 0,631) + 0,116\lambda - 0,00027$$

$$\frac{\delta Z}{\delta X_3} = -0,0422\lambda + 0,0001180(60,582\lambda + 0,369) + 0,0000822(-60,582\lambda + 0,631) + 0,116\lambda - 0,00027$$

$$\frac{\delta Z}{\delta X_4} = -0,0507\lambda + 0,0001179(60,582\lambda + 0,369) + 0,0000898(-60,582\lambda + 0,631) + 0,116\lambda - 0,00027$$

$$\frac{\delta Z}{\delta X_5} = -0,1023\lambda + 0,0001259(60,582\lambda + 0,369) + 0,0003591(-60,582\lambda + 0,631) + 0,116\lambda - 0,00027$$

$$\frac{\delta Z}{\delta \gamma} = X_1 + X_2 + X_3 + X_4 + X_5 - 1$$

$$\begin{cases} \frac{\delta Z}{\delta X_1} = 0 \Rightarrow \lambda_{c,1} = -0,00608 \\ \frac{\delta Z}{\delta X_2} = 0 \Rightarrow \lambda_{c,2} = 0,00219 \\ \frac{\delta Z}{\delta X_3} = 0 \Rightarrow \lambda_{c,3} = 0,00233 \\ \frac{\delta Z}{\delta X_4} = 0 \Rightarrow \lambda_{c,4} = 0,00256 \\ \frac{\delta Z}{\delta X_5} = 0 \Rightarrow \lambda_{c,5} = -0,00608 \end{cases}$$

The highest critical λ is equal to 0.00256, so the second solution is valid in the interval $[0,00256; 0,0104[$ and ceases to be valid when λ reaches 0.00256. At this value, the second corner portfolio (2) is obtained. The structure of this portfolio is :

$$\begin{cases} X_1 = 0,524 \\ X_2 = 0 \\ X_3 = 0 \\ X_4 = 0 \\ X_5 = 0,476 \end{cases}$$

Starting from $\lambda = 0,00256$, X_4 , which corresponds to the *Shanghai Composite index*, changes status and enters the portfolio. The matrix C and the vectors K and E then become:

$$C = \begin{pmatrix} 0,0005249 & 0,0002097 & 0,0001180 & 0,0001179 & 0,0001259 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0,0001179 & 0,0000868 & 0,0000581 & 0,0004560 & 0,0000898 & 1 \\ 0,0001259 & 0,0001285 & 0,0000822 & 0,0000898 & 0,0003591 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix};$$

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ \gamma \end{pmatrix}; K = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } E = \begin{pmatrix} 0,1406 \\ 0 \\ 0 \\ 0,0507 \\ 0,1023 \\ 0 \end{pmatrix}$$

The third solution is given by:

$$\begin{cases} X_1 = 113,416\lambda + 0,233 \\ X_2 = 0 \\ X_3 = 0 \\ X_4 = -127,829\lambda + 0,328 \\ X_5 = 14,413\lambda + 0,438 \\ \gamma = 0,094\lambda - 0,00022 \end{cases}$$

The critical λ values for the 5 considered indices are calculated by substituting the elements of the third solution into the equation $\left(\frac{\delta Z}{\delta X_{i,i=1,5}} = 0\right)$. We find:

$$\begin{cases} \frac{\delta Z}{\delta X_1} = 0 \Rightarrow \lambda_{c,1} = -0,00205 \\ \frac{\delta Z}{\delta X_2} = 0 \Rightarrow \lambda_{c,2} = 0,00207 \\ \frac{\delta Z}{\delta X_3} = 0 \Rightarrow \lambda_{c,3} = 0,00226 \\ \frac{\delta Z}{\delta X_4} = 0 \Rightarrow \lambda_{c,4} = -0,00205 \\ \frac{\delta Z}{\delta X_5} = 0 \Rightarrow \lambda_{c,5} = -0,00205 \end{cases}$$

The third solution will therefore remain valid up to ($\lambda = 0,00226$), or within the interval $[0,00226; 0,00256[$. At this value of λ , the third corner portfolio (3) corresponds, whose structure is:

$$\begin{cases} X_1 = 0,489 \\ X_2 = 0 \\ X_3 = 0 \\ X_4 = 0,039 \\ X_5 = 0,472 \end{cases}$$

Starting from ($\lambda = 0,00226$), X_3 which corresponds to the FTSE 100 index, changes status and enters the portfolio. The matrix C and the vectors K and E then become:

$$C = \begin{pmatrix} 0,0005249 & 0,0002097 & 0,0001180 & 0,0001179 & 0,0001259 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0,0001180 & 0,0001384 & 0,0001037 & 0,0000581 & 0,0000822 & 1 \\ 0,0001179 & 0,0000868 & 0,0000581 & 0,0004560 & 0,0000898 & 1 \\ 0,0001259 & 0,0001285 & 0,0000822 & 0,0000898 & 0,0003591 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix};$$

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ \gamma \end{pmatrix}; K = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } E = \begin{pmatrix} 0,1406 \\ 0 \\ 0,0422 \\ 0,0507 \\ 0,1023 \\ 0 \end{pmatrix}$$

The fourth solution is given by:

$$\begin{cases} X_1 = 240,133 - 0,052 \\ X_2 = 0 \\ X_3 = -395,385\lambda + 0,892 \\ X_4 = -27,377\lambda + 0,101 \\ X_5 = 182,629\lambda + 0,059 \\ \gamma = 0,041\lambda - 0,00009 \end{cases}$$

The critical λ values for the 5 considered indices are calculated by substituting the elements of the fourth solution into the equation $\left(\frac{\delta Z}{\delta X_{i,i=1,5}} = 0\right)$. We find:

$$\left\{ \begin{array}{l} \frac{\delta Z}{\delta X_1} = 0 \Rightarrow \lambda_{c,1} = 0,000219 \\ \frac{\delta Z}{\delta X_2} = 0 \Rightarrow \lambda_{c,2} = 0,00290 \\ \frac{\delta Z}{\delta X_3} = 0 \Rightarrow \lambda_{c,3} = 0,000218 \\ \frac{\delta Z}{\delta X_4} = 0 \Rightarrow \lambda_{c,4} = 0,000218 \\ \frac{\delta Z}{\delta X_5} = 0 \Rightarrow \lambda_{c,5} = 0,000218 \end{array} \right.$$

We note that the largest critical λ value of the fourth solution (0.00290) is greater than the critical value of the third solution (0.00226), so we stop here, and the third solution will remain valid down to ($\lambda = 0$). The following table summarizes the characteristics of the corner portfolios.

Table N°06: Composition and Performance of Corner Portfolios

Portefeuille Coin	X_1	X_2	X_3	X_4	X_5	E_{pf}^*	σ_{pf}
(1)	1,00	0,00	0,00	0,00	0,00	14,06%	1,62%
(2)	0,524	0,00	0,00	0,00	0,476	12,24%	1,20%
(3)	0,489	0,00	0,00	0,039	0,472	11,90%	1,17%

We can observe that the portfolio composition in the presence of inequality constraints has limited international diversification, where the *EURO STOXX 50* and *FTSE_100* indices do not participate in the composition of the corner portfolios ($X_2 = X_3 = 0$). This situation is explained by the high correlations between the return series of these indices and the return series *R_NASDAQ_COMPOSITE* (Table N°02 above). Furthermore, to introduce the share of the risk-free asset X_6 into the portfolio, it is sufficient to allocate the remaining risky portion ($1-X_6$) in the same proportions as the corner portfolios in Table N°06, and the new proportions remain at their corner portfolio levels. Finally, we add that all linear combinations of the corner portfolios are efficient portfolios for the investor.

3. Conclusion:

This study underscores the crucial role of international diversification in improving the performance of investment portfolios. The effectiveness of diversification, however, largely depends on the level of interdependence between global financial markets. Our empirical findings reveal that developed markets tend to be highly integrated, exhibiting strong positive correlations among their financial indices. This high integration limits the potential benefits of diversification within these markets.

Therefore, investors need to exercise prudence when relying solely on diversification strategies, as inappropriate asset allocation could lead to increased costs without corresponding gains. Nevertheless, broadening geographic exposure through international diversification has demonstrated the ability to reduce portfolio risk by nearly 30% compared to investments confined to a single market.

Additionally, the imposition of inequality constraints—such as prohibiting negative asset holdings or limiting maximum exposure—reduces the scope of diversification but still permits significant improvements in risk-adjusted returns. In conclusion, optimizing portfolio performance in today's globally interconnected markets requires a balanced investment approach that combines careful asset selection, an understanding of inter-market correlations, and practical consideration of portfolio constraints.

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