

THE VALUATION OF BARRIER OPTIONS PRICES: A METHODS REVIEW

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Abstract

This paper aims to identify existing methods to price barrier options. The question of research is to know main methods used to price barrier options. Findings of literature review based on four pricing models, which are used. In addition, there are most popular models than other ones are recently known. The method used for this paper is the bibliographic research. Barrier options are some financial products qualified derivatives. Their functioning depends on the prices evolution that the underlying assets have taken during their lifetime. Barrier options have to be knocked out or activated if the underlying price reaches the agreed limit or barrier. There are eight (08) types of european and american barrier options.

Keywords: barrier options, pricing models, tree methods, Monte Carlo methods, standard European options.

Résumé

Ce papier a pour objectif d'identifier les méthodes existantes pour évaluer les options à barrière. La question de recherche consiste à connaître les principales méthodes utilisées pour valoriser les options à barrière. Les résultats de la revue de littérature ont porté sur les quatre modèles de valorisation des options à barrière qui sont utilisés. En plus, il y a des modèles plus populaires que d'autres récemment connus. La méthode utilisée pour ce papier est la recherche bibliographique. Les options à barrière sont des produits financiers qualifiés des dérivés. Leur fonctionnement dépend de l'évolution des prix que les actifs sous-jacents ont pris durant leur durée de vie. Les options à barrière sont désactivées ou activées si le cours du sous-jacent atteint la limite convenue ou la barrière. Il y a huit (08) types d'option à barrière européenne et américaine.

Mots clés : Options à barrière, modèles d'évaluation, méthodes d'arbre, méthodes de Monte Carlo, options Européennes standards.

1.- Introduction

In the Over-Counter-Market [OTM], barrier options become extremely popular. They are the path dependent options. The existence of the option is depending upon whether the underlying asset price has touched the barrier, which is the critical value during the lifetime of the option.

In recent years, the trader uses barrier options as a useful hedging risk management instrument. In addition, barrier options have another advantage. In the trading of barrier options, the holder or buyer finds that they provide more flexibility in tailoring the returns of the portfolio. They are less expensive than standard options.

Generally, different types of barrier options are existing such as a Knock-in feature. This means that the option is activated if only the underlying spot price first hits the value called barrier. However, a knock-out feature is in the contrary case. The option is knocked-out if the underlying spot price hits the barrier. Hence, there are eight different types of barrier options. They are as follows:

- 1) Down – and – Out call option,
- 2) Up – and – Out call option,
- 3) Down – and – In call option,
- 4) Up – and – In call option,
- 5) Down – and – Out put option,
- 6) Up – and – Out put option,
- 7) Down – and – In put option,
- 8) Up – and – In put option.

The main objective of this paper is to identify the existing methods or models to price the barrier options and the actual context of the options pricing is that many articles are producing by professionals and academics. In addition, the stochastic calculus is intensively used in the mathematical formulas of barrier option pricing. The remaining of this paper is organizing as follows: the context of barrier option (2), the question of the research (3), the methodology (4), the results (5), the conclusion and discussion (6) and the further work.

2.- Context of barrier options

The barrier option can be defined as a path-dependent option. Therefore, its pay-off at maturity depends on whether or not the underlying spot price has reached some pre-defined barrier during its life. In this preliminary, we are talking about the valuation of barrier option. Some authors are known about this matter. This is the case of paper written by Boyle and Lau who studied the case of option valuation with the volatility jumps. However, Beaglehole, Kunitomo and Ikeda covered the valuation of option with double barrier.

3.- Question of research

What are the main methods or models to price the barrier options? The valuation of barrier options is the main part of our research.

4.- Methodology

Let us write that Merton was first to value the barrier options in 1973. We used for this paper the bibliographic research: books, articles, dissertations and working papers.

5.- Results

5.1. – Partial differential equations methods

Before the BS model, let us talk about the oldest model known under the name of Gaussian model from its founder a French mathematician Louis Bachelier. He received his Ph.D. on March 29, 1900 under the leadership of Pr. Henri Poincaré from the Sorbonne University. He was first who used advanced mathematics to price options. Through his thesis, titled “*Theory of speculation*”, he used a new concept known as a Brownian motion. The concept from physics is called a Wiener process. It is giving the foundation of BSM and many other models known as the financial models.

Fisher Black born in 1938 and died in 1995 due to the throat cancer received his Ph.D. in applied mathematics from the Harvard University in 1964. Meanwhile time, Myron Scholes who was born in 1941 received his Ph.D. in 1969 from the University of Chicago. In 1968, Myron Scholes joined the Massachusetts Institute of Technology (MIT) at Sloan School of Management. He took occasion to meet Black. Robert Merton was born in 1944. In addition, he got his Ph.D. in 1970 under the mentorship of Pr. Paul Samuelson. These three scientific men played a key role in the valuation formula for options. Fisher Black and Myron Scholes published the BS model in 1973. The publishing journal was Political economy. It accepted the article “*the pricing of options and Corporate liabilities*” after Eugene Fama and Merton Miller offered a second look on the article. It was in the May-June 1973 issue. Black and Scholes (1973) built the derivative pricing theory on the Geometric Brownian Motion (GBM). The price of an asset follows a GBM if it satisfies the stochastic differential equation (SDE) below:

$$dS = \mu S dt + \sigma S dW \quad (1)$$

Where:

W_t : Wiener process or Brownian motion,

μ : Drift,

σ : Volatility.

Considering the model, it was to price the european option from which the BS Partial Differential Equation is given below:

$$\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + r S \frac{\partial f}{\partial S} - \frac{\partial f}{\partial t} - rf = 0 \quad (2)$$

The solution of the equation (2) is giving the theoretical price of the European call option as follows:

$$C(S, t) = SN(d_1) - Ke^{-rT}N(d_2) \quad (3)$$

Where:

C: Call option,

S: Stock price,

K: Strike price,

r: Risk free rate,

σ : Volatility of the stock,

T: Time to maturity,

N (.): Cumulative distribution function.

From the call option, we are able to deduct its opposite, the put option.

$$P(S,t) = -S N(-d_1) + Ke^{-rT}N(-d_2) \quad (4)$$

Where:

P(S,t) : Put option

Let us explain d_1 and d_2 :

$$d_1 = [\ln(S/K) + (r + \sigma^2/2)T]\sigma\sqrt{T} \quad (5)$$

$$d_2 = [\ln(S/K) + (r - \sigma^2/2)T]\sigma\sqrt{T} \quad (6)$$

In fact, considering their impact on the Over Counter Market, some authors have written articles on the PDE method to price barrier options.

Tableau N° 1 : Articles on the PDE methods to price barrier options

Years	Authors	Country	Subject	Observation
1969	Snyder	USA	Description	Down and Out options
1973	Merton R.	USA	Closed form solution	Call down and out
1978	Brennan	USA	valuation	Real option
1983	Johnson	USA	Valuation	Put option
1983	Ewnine J.	USA	Analytical expression	Option
1983	Bergman	USA	Pricing	Put option asset
1991	Rubinstein M & Reiner	USA	Pricing	Option asset

Source: Author

The PDE method has not been the only one of methods to price barrier options. We are going to study here a specific barrier options valuation method called “tree methods”.

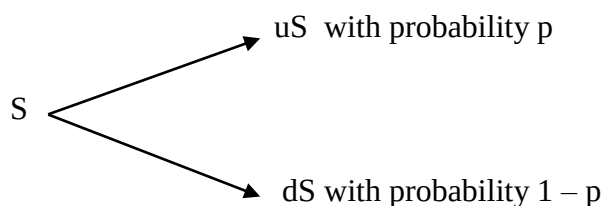
5.2. – Tree model

Barrier options are different from standard options. Tree methods are commonly used to value them. In this category of valuation methods, we have two groups of models. They are rallied in the binomial option-pricing model and the trinomial option-pricing model. What kind of definition we could give to the binomial tree model?

5.2.1. - Binomial tree model

The binomial tree model was developed in 1979 by three researchers who are John, C. Cox, Stephen A. Ross and Mark Rubinstein in their paper titled: “*Option pricing: A simplified approach*”. Valuing the option, the model has an iterative approach. There are two possible outcomes with for each interaction a move up or a move down. The probability of an up move is p , so that the probability of a down move is $1 - p$. The illustration of stock price process is as follows:

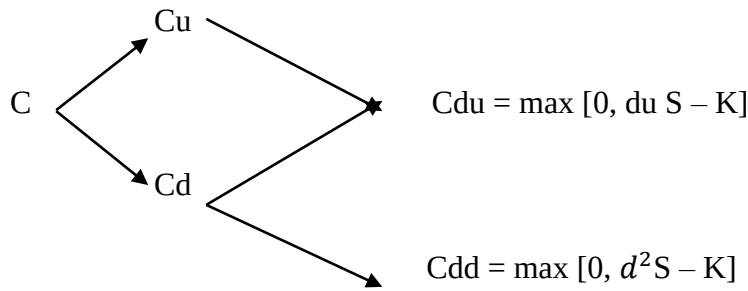
Figure N° : 1 Binomial stock tree



Source : Author

In addition, the option price process is representing as follows:

Figure N° 2 : Binomial price $C_{uu} = \max [0, u^2 S - K]$



Source : Author

Due to the limitations of the BTM, a researcher known as a professor at Waterloo University in Canada and named Phelim P. Boyle proposed (1941) in 1986 a trinomial tree model (TTM).

5. 2.2. - Trinomial Tree Model

In this model, the underlying stock price has three possible steps: an up, a down and a stable or a middle.

With:

$$u = e^{\sigma\sqrt{2\Delta t}}$$

(7)

$$d = e^{-\sigma\sqrt{2\Delta t}} = \frac{1}{u} \quad (8)$$

$$m = 1 \quad (9)$$

The transition probabilities corresponding at each node are follows:

$$p_u = \left(\frac{e^{(r-q)\Delta t/2}}{e^{\sigma\sqrt{\Delta t/2}}} - \frac{e^{-\sigma\sqrt{\Delta t/2}}}{e^{-\sigma\sqrt{\Delta t/2}}} \right)^2 \quad (10)$$

$$p_d = \left(\frac{e^{\sigma\sqrt{\Delta t/2}}}{e^{\sigma\sqrt{\Delta t/2}}} - \frac{e^{(r-q)\Delta t/2}}{e^{-\sigma\sqrt{\Delta t/2}}} \right)^2 \quad (11)$$

$$p_m = 1 - (p_u + p_d) \text{ or } 1 - p_u - p_d \quad (12)$$

Where:

u: Up, d = down, m = middle,

Δt : Length of time per step,

r: Risk-free interest rate,

σ : Volatility of the underlying,

q: Dividend yield,

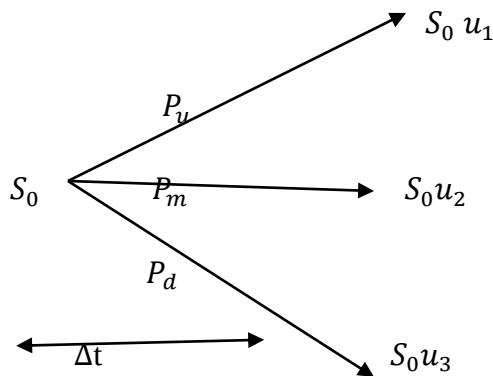
P: Probability.

p_u , p_d and p_m have to be in the interval [0, 1]. To satisfy at the above condition, the Δt must be:

$$\Delta t < 2 \frac{\sigma^2}{(r-q)^2} \quad (13)$$

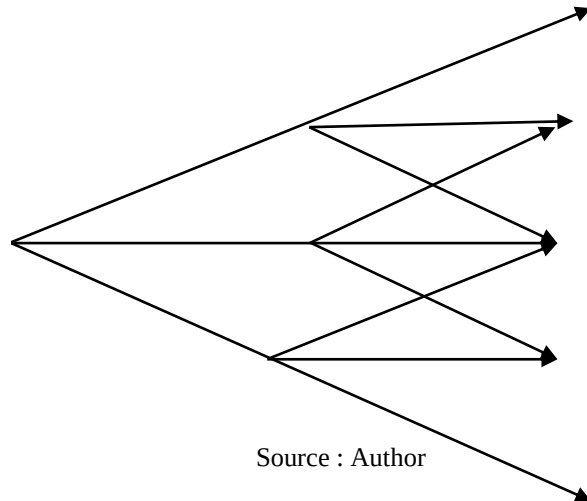
So, the tree of prices is calculated at each node. Here are examples of TTM.

Figure N° : 3 Trinomial stock tree



Source : Author

Figure N° 4 : Trinomial stock price tree



Source : Author

In this context, the Boyle's model is different from the Kamrad-Richken model developed in 1991 and the Hull-White model. We have selected authors having written some articles on the tree model.

Table N° 2 : Articles on tree model to value the barrier options

Years	Authors	Country	Subject	Observations
1994	Boyle and Lau	North Ireland	Pricing	Binomial lattice
1995	Ritchken	USA	Pricing	Trinomial lattice
1996	Cheuk and Vost	USA	Valuation	Trinomial lattice
1997	Gao	USA	Analytic High Order Trinomial	Double barrier
1999	Boyle and Tiam	North Ireland	Constant Elasticity variance	Trinomial lattice
1999	Figlewski and Gao	USA	Valuation	Trinomial lattice

Source : Author

For solving option valuation problems, the option theory in finance extended to the Monte-Carlo simulations methods.

5.3. – Monte – Carlo simulations methods

Phelim Boyle made the first application of Monte Carlo simulations methods to price option in 1977. His application made on European option. In 1996, Broadie M. and Glasserman P. used them to price Asian options. Finally, in 2001, Longstaf F. A. and Schwartz E. S. developed the practical use of these methods to price American options. However, David B. Hertz (1964) first introduced MCM in finance. As the methodology of Monte-Carlo methods, the value of the option is the result of the three steps:

- 1) Calculate potential future prices of the underlying asset,
- 2) Calculate the payoff of the option for each of the potential underlying price paths,
- 3) Discount the payoffs back to today and average then to determine the expected price.

The stock price evaluation equation following is the standard model to price any equity. The wiener process gives it.

$$S_{k\Delta t} = S_0 \exp\left(\sum_{i=1}^k \left[\left(\mu - \frac{\sigma^2}{2}\right) \Delta t + (\sigma\sqrt{\Delta t})\varepsilon_i\right]\right) \quad (14)$$

Where:

$$G = E(g(x)) = \int_a^b g(x)f(x) dx = \frac{1}{N} \sum_{i=1}^N g(x_i)$$

S_0 : Stock price today;

$S_{\Delta t}$: Stock price at a minor into the future;

Δt : Small increment of time;

μ : Expected return;

σ : Expected volatility;

ε_i : (Random) number sampled from a standard normal distribution.

The pricing of the option gives for call and put option the payoffs:

$$\text{Payoff of call option} = \text{Max}(S - K, 0) \quad (15)$$

$$\text{Payoff of put option} = \text{Max}(K - S, 0) \quad (16)$$

Where:

K: Strike price;

S: Average value of the asset price

Then,

$$G = E(g(x)) = \int_a^b g(x)f(x) dx = \frac{1}{N} \sum_{i=1}^N g(x_i) \quad (17)$$

With:

G (.): Generator function;

E: Mathematical expectation.

In our reading of literature on the Monte Carlo simulations methods, some articles and their authors are given through the table 3.

Table N° 3: Articles on Monte Carlo simulations methods

Years	Authors	Country	Subject	Observations
1977	Boyle P.	North Ireland	Valuation	Efficiency of the method
1977	Vasicek O. A.	Czech republic	Valuation	Interest Rate Model Parameters
1992	Chan Karolyi, Longstaff and Sanders	Chine (Hong Kong) ,USA	CKLS model	Factor interest rate model
1996	Andersen and Brothertom-Rotcliffe	USA	Brownian bridge	Simple constant barriers
1997	Beaglehole, Doyovig and Zhou	USA, Korea, Chine	Valuation	Simple constant barriers
1999	Baldi, Caramellino and Iovino	Italia	Sharp large deviation techniques	General diffusion process

Source : Author

Different researchers tried to find other methods to valuate barrier options that the literature in derivatives calls recent methods.

5.4. – Recent methods

They are composed with two (02) models, which are the Fast Fourier Transform Method (FFT) and the Adaptive Mesh Model (AMM). The foundation of the method is on the mathematical tool with which there are integrals on the Fourier Transform.

5. 4. 1.- Fast Fourier Transform (FFT) Methods

More researchers have used Fast Fourier Transform in their works (Cooley and Turkey, 1995; Gauss, 1805; Beagland, 1969; Strang, 1993). This method takes its origin from Fourier series having for their inventor a French mathematician Jean-Baptiste Joseph Fourier (1768-1830).

The Fourier series are including in their form a constant that we are writing as follows:

$$\begin{aligned}
 g(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n t}{T}\right) = \\
 &= \sum_{n=0}^{\infty} a_n \cos\left(\frac{2\pi n t}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n t}{T}\right)
 \end{aligned} \tag{18}$$

Where:

a, m, b and n : coefficients of the Fourier series.

The coefficients of Fourier series are having any role consisting to determine the weights for each of the sinusoids. As the generalization of the complex Fourier series, the Fourier

Transform can be defined as the function of $F_{(\omega)}$ from the function $f_{(x)}$. The function is represented by this formula:

$$F_{(\omega)} = \int_{-\infty}^{\infty} f_{(x)} e^{-i\omega x} d_{(x)} \quad (19)$$

Then, the inverse of Fourier Transform is:

$$f_{(-x)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_{(\omega)} e^{i\omega x} d_{\omega} \quad (20)$$

where :

$$i = \sqrt{-1};$$

$$\cos \theta = \cos \theta + \sin \theta.$$

Used by different researchers, Fourier Transform methods are widely recommended to value options as they have been applied also to solve a great number of problems in mathematics and physical sciences. In options prices, much in the recent literature consider the preliminaries works of Chen and Scott, 1992, Heston, 1993, Bates, 1996, Walker, 1996, Bakshi and Chen, 1997, Scott, 1997, Bakshi and Madan, 1999. Additionally, this is the case of Carr P. and Madan D. (1999) who have been first to illustrate the technique. After their seminal article "*Option valuation using the fast Fourier transform*", authors such as Duffie and al., 2000; Hubalek and al., 2006; Borovkov and Novikov, 2012; Kwok Y.Y. and al., 2012, publish many other articles.

In fact, Selby, 1983 and Buser, 1986 did have introduced first in finance to value options the Laplace Transform methods.

5.4. 2.- Laplace Transform (LT) Methods

For their sources, we consider that they are called after the invention of an integral transform by Pierre-Simon Laplace (1749-1827) who was a French mathematician and astronomer. The integral transforms a function of a real variable to a function of a complex variable. The formula of the Laplace Transform of $F(s)$ can be defined as:

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt \quad (21)$$

Where:

s: $\sigma + j\omega$:

j: $\sqrt{-1}$.

The inverse of Laplace Transform is noted : E^{-1} . S_0 . The inverse transform of $F(s)$ is the equivalent of :

$$F(s) = E^{-1}[F(s)] = E^{-1}[Ef(t)] = f(t) \quad (22)$$

In terms of the end on the Laplace Transform methods (LTM), let us write that they have been used in pricing of the exotic options like barrier options and Asian options : Geman and Yor, 1993, 1996; Pelsser, 2000; Sbuelz, 1999, 2005; Davydov and Linetsky, 2001a, 2001b; Leblanc and Scaillet, 1998 and Cathcart, 1998; Geman and Eydeland, 1995; Fu, Madan and Wang, 1998; Lipton, 1999; Fusai, 2000; Akahori, 1995; Ballotta, 2001; Ballotta and Kyprianou, 2001; Hesney et al., 1995; Chesney et al., 1997; Dassios, 1995; Hugonnier, 1999; Linetsky, 1999; Fusai and Tagliani, 2001; Fusai, 2004.

Then, it is useful to illustrate that LTM is reconverted simply from PDE into an Ordinary Differential Equation (ODE). This kind of equations is easily solvable.

However, for many problems with the valuation of exotic options, people apply for the approach of the adaptive Mesh Methods.

5.4.3.- Adaptive Mesh Methods

Figlewski and Gao, 1999 developed the AMM for exotic options. The main topic of these methods is relatively simple. Let us consider the last paper of Dong-Hyun Ahn, Stephen Figlewski and Bin Gao (1999). The AMM is presented gradually. Each step represents an equation. The base lattice is a trinomial set up to approximate the risk neutralized price process for the underlying asset. The asset price S follows the standard diffusion process.

$$d\ln S = \left(r - q - \frac{\sigma^2}{2}\right)dt + \sigma dw \quad (23)$$

Where:

S : Stock price;

r : Riskless interest rate;

q : Rate of dividend yield;

σ : Volatility;

w : Brownian motion.

It is convenient to define X , $\ln S$ and the drift $(r - q - \frac{\sigma^2}{2})$, thus, we have the equation following:

$$dX = \mu dt + \sigma dw \quad (24)$$

The process is discretized approximated by a trinomial process:

$$X_{t+k} - X_t = ak + \sigma h \text{ with the probability } p_u = \frac{k}{2h^2}, \quad ak \text{ with the probability } p_m = 1 - \frac{k}{h^2}, \\ ak - \sigma h \text{ with the probability } p_d = \frac{k}{2h^2}. \quad (25)$$

Where:

k : Discrete time step;

h : Price step;

p_u : Probability for up;

p_m : Probability for middle;

p_d : Probability for down.

The value of the option for a given asset, price and time for each node becomes:

$$V(X,t) = \exp(-rk)[p_u V(X + rk\sigma h, t+k) + p_m(X + rk, t + k) + p_d V(X + rk\sigma h, t + k)] \quad (26)$$

Where:

X: Stock price

r: $0\sqrt{k}$;

h: $\sqrt{3k}$;

$p_u = p_d = \frac{1}{6}$;

$p_m = \frac{2}{3}$.

Finally, the above model is the High Order Trinomial (HOT) model proposed by Gao B. in 1997.

Our proposal on the barrier option valuation is the use of the Backward Stochastic Differential Equations. We call the model the BSDE methods.

5.4.4. - BSDE Methods

The model to value barrier options from BSDE methods depend on the development of the backward stochastic differential equations. The pioneers in this domain are El Karoui N. (1997), Peng S. (1990), Pardoux M. (1990), Quenez M. C. (1990) and Bismut (1973). Recently the BSDEs have their applications in finance problems such as the valuation of options. As the introduction, we will start to define what the BSDE are.

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dw_t \quad Y_t = \varepsilon \quad (27)$$

In other hand, the formula can be written as:

$$dY_t = -f(t, Y_t, Z_t) dt + Z_t dW_t \quad (28)$$

$$Y_t = g(S_t)_{t \in [0, T]} \quad (29)$$

$$Y_S = E(Y_u + \int_S^u f(Y_t) dt | F_t) \quad (30)$$

Where:

f : Generator;

ε : Terminal condition.

6.- Conclusion and discussion

The evaluation methods that we have discussed in this research can be applied to value all types of barrier options: PDE methods, Tree methods, Monte Carlo Methods and recent methods in which we have proposed a new model having a basis the BSDE. Although different, all the methods have to give the same results.

7.- Further work

For future work, we will discuss about the empirical performance of each barrier options evaluation model in which we will use numerical applications.

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