

Fixed Point Theorems for Fractional Differential Inclusions Governed by Generalized Hilfer-Katugampola Operators

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Abstract—This paper investigates the existence of solutions for initial value problems involving fractional differential inclusions defined by the Hilfer-Katugampola fractional derivative, a generalization that interpolates several well-known fractional operators including Riemann-Liouville, Caputo, Hadamard, and Weyl types. We focus on fractional differential inclusions with multivalued right-hand sides and initial conditions expressed via Katugampola fractional integrals. The existence results are established using fixed point methods: the Bohnenblust-Karlin fixed point theorem is applied when the multivalued map is convex valued, while a contraction multivalued map approach based on the Covitz-Nadler fixed point theorem handles convex valued cases. An illustrative example demonstrates the applicability of the main theoretical findings, contributing novel insights to the emerging field of fractional differential inclusions under Hilfer-Katugampola operators.

Index Terms—Hilfer-Katugampola fractional derivative, fractional differential inclusions, fixed point theorem, multivalued maps, Bohnenblust-Karlin theorem, contraction multivalued map, Katugampola fractional integral, initial value problem.

I. INTRODUCTION

Recently, Oliveira et al. introduced a novel fractional derivative known as the Hilfer-Katugampola fractional derivative [2], which generalizes and unifies the classical fractional derivatives such as Katugampola, Caputo-Katugampola, Hilfer, and Hilfer-Hadamard types. This fractional operator interpolates important cases like Riemann-Liouville, Hadamard, Weyl, and Liouville derivatives, offering a comprehensive framework for fractional calculus [5], [6], [7], [8], [9]. Motivated by the foundational work in [10], [11], [12], [13], [14], which established existence and uniqueness for nonlinear fractional differential equations involving the Hilfer-Katugampola derivative with initial conditions, this paper extends the investigation to fractional differential inclusions.

$${}^{\mathfrak{S}}\mathcal{D}_{a+}^{\lambda, \flat} \eta(\iota) \in F(\iota, \eta(\iota)), \quad \iota \in \mathcal{J} := [a, b], \quad (1)$$

$$({}^{\mathfrak{S}}I^{1-\kappa} \eta)(a) = c, \quad c \in \mathbb{R}, \quad \kappa = \lambda + \flat(1 - \lambda), \quad (2)$$

where the parameters satisfy $\lambda \in (0, 1)$, $\flat \in [0, 1]$, and $\mathfrak{S} > 0$. Let $F : \mathcal{J} \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ be a multivalued mapping, where $\mathcal{P}(\mathbb{R})$ represents the family of all nonempty subsets of $\mathbb{R}$. The operators ${}^{\mathfrak{S}}\mathcal{D}_{a+}^{\lambda, \flat}$ and ${}^{\mathfrak{S}}I^{1-\kappa}$ refer, respectively, to the Hilfer-Katugampola fractional derivative of order λ and type $\flat$, and to the Katugampola fractional integral of order $1 - \kappa$, with $a > 0$.

Our analysis focuses on the existence of solutions to an initial value problem formulated in terms of fractional differential

inclusions driven by the Hilfer-Katugampola operator, where the right-hand side involves a multivalued mapping. The considered model is expressed as a fractional differential inclusion containing this derivative together with a fractional integral initial condition. The analysis is based on the Bohnenblust-Karlin fixed point theorem applied to convex-valued multivalued maps, and on the Covitz-Nadler fixed point theorem in the framework of multivalued contractions when convex-valued maps are involved.

The organization of this paper is as follows. Section 2 recalls necessary preliminaries and definitions concerning fractional calculus and multivalued operators. Section 3 presents the main results on the existence of solutions, showing the applicability of fixed point methods to the proposed inclusion. Section 4 provides an illustrative example. This research extends the study of fractional differential inclusions involving the Hilfer-Katugampola operator and enhances the link between generalized fractional operators and fixed point theory.

II. PRELIMINARIES

In this part, we present the notations, fundamental definitions, and basic results that will be utilized throughout the manuscript. Consider $C(\mathcal{J}, \mathbb{R})$ as the Banach space formed by all real-valued continuous functions on the interval $\mathcal{J}$, endowed with the norm defined by

$$\|\eta\|_C = \max_{\iota \in \mathcal{J}} |\eta(\iota)|.$$

The space $L^1(\mathcal{J}, \mathbb{R})$ is composed of all Lebesgue integrable real-valued functions on $\mathcal{J}$, equipped with the norm

$$\|\eta\|_{L^1} = \int_a^b |\eta(\iota)| d\iota.$$

Finally, $L^\infty(\mathcal{J}, \mathbb{R})$ denotes the Banach space of essentially bounded measurable functions mapping $\mathcal{J}$ into $\mathbb{R}$, where the norm is given by

$$\|\eta\|_{L^\infty} = \inf\{\kappa > 0 : |\eta(\iota)| \leq \kappa \text{ for almost every } \iota \in \mathcal{J}\}.$$

We proceed by presenting certain fundamental results and characteristics related to fractional calculus.

Definition 1: For $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) > 0$, the left-sided fractional integral introduced by Katugampola is defined for $\zeta > 0$ as

$${}^{\mathfrak{S}}I^\lambda h(\zeta) = \frac{\mathfrak{S}^{1-\lambda}}{\Gamma(\lambda)} \int_0^\zeta (\zeta^\mathfrak{S} - j^\mathfrak{S})^{\lambda-1} j^{\mathfrak{S}-1} h(j) dj, \quad (3)$$

provided that the integral converges.

Definition 2: The left-sided fractional derivative introduced by Katugampola, which is associated with the fractional integral given in (3), is defined for $\zeta > 0$ by

$${}^{\mathfrak{S}}\mathcal{D}^{\lambda}h(\zeta) = \frac{\mathfrak{S}^{\lambda-n+1} \left(\zeta^{1-\mathfrak{S}} \frac{d}{d\zeta} \right)^n}{\Gamma(n-\lambda)} \int_0^{\zeta} (\zeta^{\mathfrak{S}} - j^{\mathfrak{S}})^{n-\lambda-1} j^{\mathfrak{S}-1} h(j) dj,$$

where $n = \lfloor \lambda \rfloor + 1$, assuming the integral is well-defined.

Definition 3: . The Caputo-Katugampola fractional derivative $({}^{\mathfrak{S}}\mathcal{D}_{a+}^{\lambda}h)(\iota)$ is defined via the above Katugampola fractional derivative as follows

$$({}^{\mathfrak{S}}\mathcal{D}_{a+}^{\lambda}h)(\iota) = ({}^{\mathfrak{S}}\mathcal{D}_{a+}^{\lambda}[h(j) - h(a)])(\iota), \quad 0 < \lambda < 1$$

Definition 4: For order $\lambda \in (0, 1)$ and type $b \in [0, 1]$, the Hilfer-Katugampola fractional derivative with respect to ι is given by

$${}^{\mathfrak{S}}\mathcal{D}_{a+}^{\lambda,b}h(\iota) = \left({}^{\mathfrak{S}}I_{a+}^{b(1-\lambda)} \delta_{\mathfrak{S}} {}^{\mathfrak{S}}I_{a+}^{(1-b)(1-\lambda)} h \right)(\iota), \quad \mathfrak{S} > 0. \quad (4)$$

for all functions h for which the right-hand side is well-defined. Note that $0 < (1-b)(1-\lambda) < 1$.

Remark 5: The operator ${}^{\mathfrak{S}}\mathcal{D}_{a+}^{\lambda,b}$ can be represented in terms of the Katugampola fractional derivative as

$${}^{\mathfrak{S}}\mathcal{D}_{a+}^{\lambda,b} = {}^{\mathfrak{S}}I_{a+}^{b(1-\lambda)} \delta_{\mathfrak{S}} {}^{\mathfrak{S}}I_{a+}^{1-\lambda} = {}^{\mathfrak{S}}I_{a+}^{b(1-\lambda)} \mathcal{D}_{a+}^{\lambda},$$

with $\lambda = \lambda + b - \lambda b$.

Lemma 6 (Semigroup property): Let $\lambda, b > 0$, $1 \leq p \leq \infty$, and $0 < a < b < \infty$, where $\mathfrak{S}, c \in \mathbb{R}$ with $\mathfrak{S} \geq c$. Then, for every $h \in \zeta_c^p(a, b)$, the following semigroup relation holds:

$$\left({}^{\mathfrak{S}}I_{a+}^{\lambda} {}^{\mathfrak{S}}I_{a+}^b h \right)(\iota) = \left({}^{\mathfrak{S}}I_{a+}^{\lambda+b} h \right)(\iota).$$

Lemma 7: Given $\iota > a$, and the fractional integral and derivative operators ${}^{\mathfrak{S}}I_{a+}^{\lambda}$ and ${}^{\mathfrak{S}}\mathcal{D}_{a+}^{\lambda}$, the following identities are satisfied:

(i) For $\lambda \geq 0$, $\hbar > 0$,

$$\left({}^{\mathfrak{S}}I_{a+}^{\lambda} \left[\left(\frac{j^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\hbar} \right] \right)(\iota) = \frac{\Gamma(\hbar + 1)}{\Gamma(\hbar + \lambda + 1)} \left(\frac{\iota^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\hbar + \lambda}.$$

(ii) When $\hbar = 0$,

$$\left({}^{\mathfrak{S}}I_{a+}^{\lambda} 1 \right)(\iota) = \frac{\left(\frac{\iota^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda}}{\Gamma(\lambda + 1)}, \quad \lambda \geq 0.$$

(iii) For $0 < \lambda < 1$,

$$\left({}^{\mathfrak{S}}\mathcal{D}_{a+}^{\lambda} \left(\frac{j^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda-1} \right)(\iota) = 0.$$

A. Multivalued Operators

Suppose that $\aleph$ and Y are Banach spaces. We study several families of subsets of $\aleph$:

$$\mathcal{P}(\aleph) := \{ K \subseteq \aleph ; K \neq \emptyset \},$$

$$\mathcal{P}_{cl}(\aleph) := \{ K \in \mathcal{P}(\aleph) ; K \text{ is closed} \},$$

$$\mathcal{P}_b(\aleph) := \{ K \in \mathcal{P}(\aleph) ; K \text{ is bounded} \},$$

$$\mathcal{P}_{cv}(\aleph) := \{ K \in \mathcal{P}(\aleph) ; K \text{ is convex} \},$$

$$\mathcal{P}_{cp}(\aleph) := \{ K \in \mathcal{P}(\aleph) ; K \text{ is compact} \}.$$

A set-valued function $G : \aleph \rightarrow \mathcal{P}(Y)$ is termed compact if the union $G(\aleph) := \bigcup_{\zeta \in \aleph} G(\zeta)$ forms a compact set in Y . We say that G has convex, closed, or compact values if for each $\zeta \in \aleph$, the set $G(\zeta)$ possesses the corresponding property. Moreover, the operator G is said to be bounded on bounded subsets whenever, for each bounded set $\mathcal{B} \subset \aleph$, its image $G(\mathcal{B})$ stays bounded in Y .

The map G is upper semicontinuous (u.s.c.) at a point $\zeta_0 \in \aleph$ if $G(\zeta_0)$ is nonempty and closed, and whenever an open set N contains $G(\zeta_0)$, there exists an open neighborhood N_0 of ζ_0 satisfying $G(N_0) \subseteq N$.

If the operator G sends every bounded subset of $\aleph$ into a relatively compact subset of Y , it is referred to as completely continuous.

Moreover, when G is completely continuous with nonempty compact values, it is upper semicontinuous precisely when its graph is closed; in other words, whenever sequences $\zeta_n \rightarrow \zeta_*$ and $\eta_n \rightarrow \eta_*$ fulfill the condition $\eta_n \in G(\zeta_n)$, it follows that $\eta_* \in G(\zeta_*)$ holds.

We denote by $\mathcal{P}_{b,cl,cv}(\aleph)$ the family consisting of all nonempty subsets of $\aleph$ that are bounded, closed, and convex.

A multivalued operator $G : \mathcal{J} \rightarrow \mathcal{P}_{cl}(\aleph)$ is said to be measurable whenever, for every fixed $\eta \in \aleph$, the mapping $\iota \mapsto \text{dist}(\eta, G(\iota))$ is measurable on $\mathcal{J}$.

If $\aleph \subseteq Y$, a fixed point of G is a point $\eta \in \aleph$ satisfying $\eta \in G(\eta)$. The collection of all such elements is denoted by $\text{Fix}(G)$.

For each $\eta \in \aleph$, the quantity $\|G(\eta)\|_{\mathcal{P}}$ is defined as

$$\|G(\eta)\|_{\mathcal{P}} = \sup \{ \|\zeta\| ; \zeta \in G(\eta) \}.$$

Within the particular case where the values lie in $\mathbb{R}$, a multivalued mapping A mapping $G : \mathcal{J} \rightarrow \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable whenever, for each fixed $\eta \in \mathbb{R}$, the function

$$\iota \mapsto d(\eta, G(\iota)) := \inf_{z \in G(\iota)} |\eta - z|$$

is measurable on $\mathcal{J}$.

Definition 8: A multivalued function $F : \mathcal{J} \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is called an L^1 -Carathéodory map if it satisfies the following conditions:

- (i) For each fixed $\eta \in \mathbb{R}$, the function $\iota \mapsto F(\iota, \eta)$ is measurable;
- (ii) For almost every $\iota \in \mathcal{J}$, the correspondence $\eta \mapsto F(\iota, \eta)$ is upper semicontinuous;
- (iii) For any positive real number q , one can find a function $\varphi_q \in L^1(\mathcal{J}, \mathbb{R}_+)$ satisfying

$$\|F(\iota, \eta)\|_{\mathcal{P}} = \sup \{ |f| ; f \in F(\iota, \eta) \} \leq \varphi_q(\iota),$$

for almost every $\iota \in \mathcal{J}$ and all η such that $|\eta| \leq q$.

A multivalued map F is called a Carathéodory map if it meets conditions (i) and (ii) as previously stated.

For any function $\eta \in C_{\kappa, \mathfrak{S}}(\mathcal{J}, \mathbb{R})$, the associated set of selections of F is defined by

$$S_{F, \eta} = \{g \in L^1(\mathcal{J}, \mathbb{R}) : g(\iota) \in F(\iota, \eta(\iota)) \text{ for almost every } \iota \in \mathcal{J}\}.$$

Let $(\mathfrak{N}, d)$ be a metric space induced by the norm $|\cdot|$ on $\mathfrak{N}$. Define the Hausdorff distance $H_d : \mathcal{P}(\mathfrak{N}) \times \mathcal{P}(\mathfrak{N}) \rightarrow \mathbb{R}_+ \cup \{\infty\}$ by

$$H_d(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\},$$

where

$$d(A, b) = \inf_{a \in A} d(a, b), \quad \text{and} \quad d(a, B) = \inf_{b \in B} d(a, b).$$

It follows that $(\mathcal{P}_{b, cl}(\mathfrak{N}), H_d)$ constitutes a metric space, whereas $(\mathcal{P}_{cl}(\mathfrak{N}), H_d)$ represents a generalized metric space.

Definition 9: A multivalued operator $N : \mathfrak{N} \rightarrow \mathcal{P}_{cl}(\mathfrak{N})$ is described as follows:

(a) The mapping is said to be γ -Lipschitz whenever there exists a real constant $\gamma > 0$ satisfying, for all $\zeta, \eta \in \mathfrak{N}$,

$$\gamma H_d(N(\zeta), N(\eta)) \leq \gamma d(\zeta, \eta).$$

(b) The operator N is called a contraction if it satisfies the γ -Lipschitz condition with $\gamma < 1$.

The lemma below will play a key role in the following developments.

Lemma 10 (Bohnenblust–Karlin, see [3]): Let $\mathfrak{N}$ denote a Banach space, and take $K \in \mathcal{P}_{cl, c}(\mathfrak{N})$. Suppose that the mapping $G : K \rightarrow \mathcal{P}_{cl, c}(K)$ is upper semicontinuous, and that the set $G(K)$ has a relatively compact image in $\mathfrak{N}$. Under these conditions, the operator G admits at least one fixed point in K .

Lemma 11: Consider a metric space $(\mathfrak{N}, d)$ that is complete. If $N : \mathfrak{N} \rightarrow \mathcal{P}_{cl}(\mathfrak{N})$ is a contraction, it follows that $\text{Fix } N \neq \emptyset$.

For additional information concerning multivalued mappings, one may refer to the monographs by Deimling [4].

III. MAIN RESULTS

Consider the weighted Banach space,

$$C_{\kappa, \mathfrak{S}}(\mathcal{J}, \mathbb{R}) = \{u : \mathcal{J}^* := (a, b] \rightarrow \mathbb{R} : \left(\frac{\iota^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}}\right)^{1-\kappa} u(\iota) \in C(\mathcal{J}^*)\},$$

$0 < \kappa \leq 1$, with the norm

$$\|u\|_{C_{\kappa, \mathfrak{S}}} = \left\| \left(\frac{\iota^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}}\right)^{1-\kappa} u(\iota) \right\|_C = \max_{\iota \in \mathcal{J}} \left| \left(\frac{\iota^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}}\right)^{1-\kappa} u(\iota) \right|.$$

We begin by specifying the notion of a solution for the problem (1)-(2).

Definition 12: A function $\eta \in C_{\kappa, \mathfrak{S}}(\mathcal{J}, \mathbb{R})$ is called a solution to the problem (1)-(2) whenever there exists a function $g \in L^1(\mathcal{J}, \mathbb{R})$ fulfilling $g(\iota) \in F(\iota, \eta(\iota))$ for almost all $\iota \in \mathcal{J}$, and the element η verifies the fractional differential equation

$${}^{\mathfrak{S}}\mathcal{D}_{a+}^{\lambda, \flat} \eta(\iota) = g(\iota), \quad \iota \in \mathcal{J},$$

together with the initial condition

$${}^{\mathfrak{S}}I_{a+}^{1-\kappa} \eta(a) = c.$$

To demonstrate the presence of at least one solution to the system (1)-(2), we rely on the next essential lemma.

Lemma 13 (see [2], [1]): Let $\kappa = \lambda + \flat(1 - \lambda)$ with $0 < \lambda < 1$, $0 \leq \flat \leq 1$, and $\mathfrak{S} > 0$. For any $g \in C_{\kappa, \mathfrak{S}}(\mathcal{J}, \mathbb{R})$, A function η fulfills the fractional integral relation

$$\eta(\iota) = \frac{c}{\Gamma(\kappa)} \left(\frac{\iota^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\kappa-1} + \int_a^{\iota} j^{\mathfrak{S}-1} \left(\frac{\iota^{\mathfrak{S}} - j^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda-1} \frac{g(j)}{\Gamma(\lambda)} dj,$$

if and only if η represents a solution to the corresponding fractional initial value formulation.

$${}^{\mathfrak{S}}\mathcal{D}^{\lambda, \flat} \eta(\iota) = g(\iota), \quad \iota \in \mathcal{J},$$

$$({}^{\mathfrak{S}}I^{1-\kappa} \eta)(a) = c, \quad c \in \mathbb{R},$$

where $\kappa = \lambda + \flat(1 - \lambda)$.

Our focus is to demonstrate the presence of at least one solution to the system (1)-(2), assuming that the right-hand side takes values in convex sets. Accordingly, we consider F as a multivalued map with compact and convex values.

We impose the following conditions:

(H1) The mapping $F : \mathcal{J} \times \mathbb{R} \rightarrow \mathcal{P}_{cv}(\mathbb{R})$ is a Carathéodory multivalued map.

(H2) Assume there exists a continuous mapping $p : \mathcal{J} \rightarrow \mathbb{R}^+$ such that, for each $\iota \in \mathcal{J}$ and every $\eta \in \mathbb{R}$,

$$\|F(\iota, \eta)\|_{\mathcal{P}} = \sup\{|g| : g \in F(\iota, \eta)\} \leq \frac{p(\iota)}{1 + |\eta|} |\eta|.$$

Define $p^* := \sup_{\iota \in \mathcal{J}} p(\iota)$.

(H3) Suppose there exists a bounded function $\tilde{\ell} \in L^\infty(\mathcal{J}, \mathbb{R}^+)$ satisfying, for almost all $\iota \in \mathcal{J}$ and any $\eta, \bar{\eta} \in \mathbb{R}$,

$$H_d(F(\iota, \eta), F(\iota, \bar{\eta})) \leq \tilde{\ell}(\iota) |\eta - \bar{\eta}|,$$

together with

$$d(0, F(\iota, 0)) \leq \tilde{\ell}(\iota).$$

The initial main result relies on the Bohnenblust-Karlin fixed point theorem.

Theorem 14: Suppose the conditions (H1) through (H3) hold true. Then the initial value problem (1)-(2) admits at least one solution defined on $\mathcal{J}$.

Proof. We reformulate the problem (1)-(2) as a fixed point problem. Define the multivalued operator

$$\iota : C_{\kappa, \mathfrak{S}}(\mathcal{J}, \mathbb{R}) \rightarrow \mathcal{P}(C_{\kappa, \mathfrak{S}}(\mathcal{J}, \mathbb{R}))$$

by setting:

$$(\iota\eta)(\iota) = \{h \in C_{\kappa, \mathfrak{S}}(\mathcal{J}, \mathbb{R}) : h(\iota) = \frac{c}{\Gamma(\kappa)} \left(\frac{\iota^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\kappa-1} + \int_a^{\iota} j^{\mathfrak{S}-1} \left(\frac{\iota^{\mathfrak{S}} - j^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda-1} \frac{g(j)}{\Gamma(\lambda)} dj\}.$$

It is evident that any fixed point η of the operator equation $(\iota\eta)(\iota) = \eta(\iota)$ corresponds to a solution of the problem (1)-(2).

For each $\eta \in C_{\kappa, \mathfrak{S}}(\mathcal{J}, \mathbb{R})$ and every $\iota \in \mathcal{J}$, we obtain

$$\begin{aligned} & \left| \left(\frac{\iota^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa} (\eta)(\iota) \right| \\ & \leq \frac{|c|}{\Gamma(\kappa)} + \left(\frac{\iota^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa} \int_a^{\iota} j^{\mathfrak{S}-1} \left(\frac{\iota^{\mathfrak{S}} - j^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda-1} \frac{|g(j)|}{\Gamma(\lambda)} dj \\ & \leq \frac{|c|}{\Gamma(\kappa)} + \frac{\left(\frac{\iota^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa}}{\Gamma(\lambda)} \int_a^{\iota} j^{\mathfrak{S}-1} \left(\frac{\iota^{\mathfrak{S}} - j^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda-1} p(j) dj \\ & \leq \frac{|c|}{\Gamma(\kappa)} + \frac{p^*}{\Gamma(\lambda)} \left(\frac{\iota^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa} \int_a^{\iota} j^{\mathfrak{S}-1} \left(\frac{\iota^{\mathfrak{S}} - j^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda-1} dj \\ & \leq \frac{|c|}{\Gamma(\kappa)} + \frac{p^*}{\Gamma(\lambda+1)} \left(\frac{\iota^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa} \left(\frac{\iota^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda} \\ & \leq \frac{|c|}{\Gamma(\kappa)} + \frac{p^*}{\Gamma(\lambda+1)} \left(\frac{b^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda+1-\kappa} \end{aligned}$$

Thus

$$\|\eta\|_C \leq \frac{|c|}{\Gamma(\kappa)} + \frac{p^*}{\Gamma(\lambda+1)} \left(\frac{T^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda+1-\kappa} := \wp \quad (5)$$

Therefore, the operator ι maps the ball $B_{\wp} = B(0, \wp) = \{\eta \in C_{\kappa, \mathfrak{S}} : \|\eta\|_C \leq \wp\}$ into the same set.

We aim to demonstrate that ι meets the conditions required by the Bohnenblust–Karlin fixed point theorem.

The argument will be developed through a sequence of steps.

Step 1: For each $\eta \in B_{\wp}$, the set $\iota(\eta)$ is convex.

To see this, suppose $h_1, h_2 \in \iota(\eta)$. Then there exist functions $g_1, g_2 \in S_{F, \eta}$ such that for each $\iota \in \mathcal{J}$, and $i = 1, 2$,

$$h_i(\iota) = \frac{c}{\Gamma(\kappa)} \left(\frac{\iota^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\kappa-1} + \int_a^{\iota} j^{\mathfrak{S}-1} \left(\frac{\iota^{\mathfrak{S}} - j^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda-1} \frac{g_i(j)}{\Gamma(\lambda)} dj$$

Take any $d \in [0, 1]$. Then, for every $\iota \in \mathcal{J}$, it holds that

$$\begin{aligned} & (dh_1 + (1-d)h_2)(\iota) = \\ & \int_a^{\iota} j^{\mathfrak{S}-1} \left(\frac{\iota^{\mathfrak{S}} - j^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda-1} \frac{[dg_1(j) + (1-d)g_2(j)]}{\Gamma(\lambda)} dj. \end{aligned}$$

Because the set $S_{F, \eta}$ is convex, due to the convexity of the values of F , it follows that

$$dh_1 + (1-d)h_2 \in \iota(\eta).$$

Step 2: The set $\iota(B_{\wp})$ is uniformly bounded.

Since $\iota(B_{\wp}) \subseteq B_{\wp}$, and $B_{\wp}$ itself is bounded, it immediately follows that $\iota(B_{\wp})$ is uniformly bounded.

Step 3: The family $\iota(B_{\wp})$ is equicontinuous.

Take $\iota_1, \iota_2 \in \mathcal{J}$ with $\iota_1 < \iota_2$ and fix $\eta \in B_{\wp}$. For each $h \in \iota(\eta)$ and any $\iota \in \mathcal{J}$, using assumption (H2), we obtain

$$\begin{aligned} & \left| \left(\frac{\iota_2^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa} h(\iota_2) - \left(\frac{\iota_1^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa} h(\iota_1) \right| \\ & \leq \left| \frac{\left(\frac{\iota_2^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa}}{\Gamma(\lambda)} \int_a^{\iota_2} j^{\mathfrak{S}-1} \left(\frac{\iota_2^{\mathfrak{S}} - j^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda-1} g(j) dj \right. \\ & \quad \left. - \frac{\left(\frac{\iota_1^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa}}{\Gamma(\lambda)} \int_a^{\iota_1} j^{\mathfrak{S}-1} \left(\frac{\iota_1^{\mathfrak{S}} - j^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda-1} g(j) dj \right| \\ & \leq \frac{\left(\frac{\iota_2^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa}}{\Gamma(\lambda)} \int_{\iota_1}^{\iota_2} j^{\mathfrak{S}-1} \left(\frac{\iota_2^{\mathfrak{S}} - j^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda-1} |g(j)| dj \\ & \quad + \int_a^{\iota_1} \left| \left(\frac{\iota_2^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa} j^{\mathfrak{S}-1} \left(\frac{\iota_2^{\mathfrak{S}} - j^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda-1} \right. \\ & \quad \left. - \left(\frac{\iota_1^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa} j^{\mathfrak{S}-1} \left(\frac{\iota_1^{\mathfrak{S}} - j^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda-1} \right| \frac{|g(j)|}{\Gamma(\lambda)} dj \\ & \leq \frac{\left(\frac{\iota_2^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa}}{\Gamma(\lambda)} \int_{\iota_1}^{\iota_2} j^{\mathfrak{S}-1} \left(\frac{\iota_2^{\mathfrak{S}} - j^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda-1} p(j) dj \\ & \quad + \int_a^{\iota_1} \left| \left(\frac{\iota_2^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa} j^{\mathfrak{S}-1} \left(\frac{\iota_2^{\mathfrak{S}} - j^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda-1} \right. \\ & \quad \left. - \left(\frac{\iota_1^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa} j^{\mathfrak{S}-1} \left(\frac{\iota_1^{\mathfrak{S}} - j^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda-1} \right| \frac{p(j)}{\Gamma(\lambda)} dj \end{aligned}$$

As a consequence of the continuity of the function p , we deduce

$$\begin{aligned} & \left| \left(\frac{\iota_2^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa} (h\eta)(\iota_2) - \left(\frac{\iota_1^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa} (h\eta)(\iota_1) \right| \\ & \leq \frac{p^*}{\Gamma(\lambda+1)} \left(\frac{T^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa+\lambda} \left(\frac{\iota_2^{\mathfrak{S}} - \iota_1^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda} \\ & \quad + \frac{p^*}{\Gamma(\lambda)} \left| \int_a^{\iota_1} \left(\frac{\iota_2^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa} j^{\mathfrak{S}-1} \left(\frac{\iota_2^{\mathfrak{S}} - j^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda-1} \right. \\ & \quad \left. - \left(\frac{\iota_1^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa} j^{\mathfrak{S}-1} \left(\frac{\iota_1^{\mathfrak{S}} - j^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda-1} \right| dj \end{aligned}$$

When $\iota_1 \rightarrow \iota_2$, the right-hand side of the inequality above approaches zero.

Combining Steps 1 through 3 and applying the Arzela–Ascoli theorem, it follows that the operator $\iota : C_{\kappa, \mathfrak{S}}(\mathcal{J}, \mathbb{R}) \rightarrow \mathcal{P}(C_{\kappa, \mathfrak{S}}(\mathcal{J}, \mathbb{R}))$ is completely continuous.

Step 4: The operator ι possesses a closed graph.

Suppose sequences $\eta_n \rightarrow \eta_*$ and $h_n \in \iota(\eta_n)$ satisfy $h_n \rightarrow h_*$. We must establish that $h_* \in \iota(\eta_*)$. Since $h_n \in \iota(\eta_n)$, there exist functions $g_n \in S_{F, \eta_n}$ such that for each $\iota \in \mathcal{J}$,

$$h_n(\iota) = \frac{c}{\Gamma(\kappa)} \left(\frac{\iota^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\kappa-1} + \int_a^{\iota} j^{\mathfrak{S}-1} \left(\frac{\iota^{\mathfrak{S}} - j^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda-1} \frac{g_n(j)}{\Gamma(\lambda)} dj.$$

It remains to demonstrate the existence of a function $g_*(\iota) \in S_{F, \eta_*}$ such that for every $\iota \in \mathcal{J}$,

$$h_*(\iota) = \frac{c}{\Gamma(\kappa)} \left(\frac{\iota^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\kappa-1} + \int_a^{\iota} j^{\mathfrak{S}-1} \left(\frac{\iota^{\mathfrak{S}} - j^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda-1} \frac{g_*(j)}{\Gamma(\lambda)} dj.$$

Since $F(\iota, \cdot)$ is upper semicontinuous, for any $\epsilon > 0$, there exists an index $n_0(\epsilon) \geq 0$ such that for all $n \geq n_0$,

$$g_n(\iota) \in F(\iota, \eta_n(\iota)) \subseteq F(\iota, \eta_*(\iota)) + \epsilon B(0, 1) \quad \text{for almost every } \iota \in \mathcal{J}.$$

Because the values of F are compact, there is a subsequence (g_{n_m}) satisfying

$$g_{n_m} \rightarrow g_* \quad \text{as } m \rightarrow \infty,$$

with

$$g_*(\iota) \in F(\iota, \eta_*(\iota)) \quad \text{for almost every } \iota \in \mathcal{J}.$$

For any $\hbar \in F(\iota, \eta_*(\iota))$, we have the inequality

$$|g_{n_m}(\iota) - g_*(\iota)| \leq |g_{n_m}(\iota) - \hbar| + |\hbar - g_*(\iota)|,$$

and so

$$|g_{n_m}(\iota) - g_*(\iota)| \leq d(g_{n_m}(\iota), F(\iota, \eta_*(\iota))).$$

Using a similar argument by switching the positions of g_{n_m} and g_* , we obtain

$$\begin{aligned} |g_{n_m}(\iota) - g_*(\iota)| &\leq H_d(F(\iota, \eta_{n_m}(\iota)), F(\iota, \eta_*(\iota))) \\ &\leq \tilde{\ell} |\eta_{n_m} - \eta_*|. \end{aligned}$$

Hence,

$$\begin{aligned} &\left| \left(\frac{\iota^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa} (\hbar_{n_m}(\iota) - \hbar_*(\iota)) \right| \\ &\leq \frac{\left(\frac{\iota^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{1-\kappa}}{\Gamma(\lambda)} \int_a^\iota j^{\mathfrak{S}-1} \left(\frac{\iota^{\mathfrak{S}} - j^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda-1} \\ &\quad \times |g_{n_m}(j) - g_*(j)| dj \|\hbar_{n_m} - \hbar_*\|_C \\ &\leq \frac{\ell^*}{\Gamma(\lambda+1)} \left(\frac{\iota^{\mathfrak{S}} - a^{\mathfrak{S}}}{\mathfrak{S}} \right)^{\lambda+1-\kappa} \|\eta_{n_m} - \eta_*\|_C. \end{aligned}$$

where $\ell^* = \|\tilde{\ell}\|_{L^\infty}$

Then

$$\|\hbar_{n_m} - \hbar_*\|_C \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Consequently, applying Lemma 10, we conclude that ι admits a fixed point η in $B_\varphi \subset C_{\kappa, \mathfrak{S}}(\mathcal{J}, \mathbb{R})$, which serves as a solution to the initial value problem (1)-(2). This concludes the proof of the theorem.

IV. EXAMPLE

To illustrate our results, consider the following fractional initial value problem:

$${}^{\mathfrak{S}}\mathcal{D}_{a^+}^{\frac{1}{2}, \frac{1}{2}} \eta(\iota) \in F(\iota, \eta(\iota)), \quad \iota \in \mathcal{J} := [1, 2], \quad (6)$$

$$\left({}^{\mathfrak{S}}I^{\frac{1}{4}} \eta \right) (1) = 0, \quad (7)$$

where $\lambda = \frac{1}{2}$, $\mathfrak{b} = \frac{1}{2}$, $\mathfrak{S} > 0$, and $\kappa = \frac{3}{4}$.

Define the multivalued map

$$F(\iota, \eta) = \{g \in \mathbb{R} : g_1(\iota, \eta) \leq g \leq g_2(\iota, \eta)\},$$

where the functions $g_1, g_2 : \mathcal{J} \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable in ι .

We suppose that for each fixed $\iota \in \mathcal{J}$, the function $g_1(\iota, \cdot)$ is lower semicontinuous; that is, the set $\{\eta \in \mathbb{R} : g_1(\iota, \eta) > \mu\}$ is open for every $\mu \in \mathbb{R}$. Similarly, $g_2(\iota, \cdot)$ is upper semicontinuous for each ι , meaning the set $\{\eta \in \mathbb{R} : g_2(\iota, \eta) < \mu\}$ is open for all $\mu \in \mathbb{R}$.

Assume there exists a continuous function $p \in C(\mathcal{J}, \mathbb{R}^+)$ such that

$$\max(|g_1(\iota, \eta)|, |g_2(\iota, \eta)|) \leq \frac{\sin(\iota-1)}{1+\sqrt{\iota-1}} |\eta|, \quad \iota \in \mathcal{J}.$$

Hence, hypothesis (H2) is fulfilled by setting

$$p(\iota) = \frac{\sin(\iota-1)}{1+\sqrt{\iota-1}}, \quad \iota \in \mathcal{J}.$$

From these properties, the mapping F is compact- and convex-valued as well as upper semicontinuous (see [4]). Since all the assumptions of Theorem 14 are met, the initial value problem (6)-(7) admits at least one solution η on $\mathcal{J}$.

CONCLUSION

The article establishes the existence of solutions for fractional differential inclusions governed by the Hilfer-Katugampola fractional derivative, a general operator unifying several classical fractional derivatives. By formulating the problem with multivalued right-hand sides and initial fractional integral conditions, and applying fixed point theorems specifically the Bohnenblust-Karlin theorem this work rigorously extends the theory of fractional differential inclusions in a novel direction. The assumptions imposed on the multivalued map ensure the operator is well defined, convex-valued, and upper semicontinuous, enabling the demonstration of at least one solution.

This contribution enriches the fractional calculus literature by linking generalized fractional operators with multivalued analysis and set-valued fixed point methods, providing a foundational framework for future studies in fractional inclusions involving Hilfer-Katugampola derivatives. Moreover, the illustrative example confirms the applicability of the theoretical results. Future work may explore uniqueness, continuous dependence, and numerical approximation schemes for such fractional inclusions, as well as potential applications in modeling complex systems exhibiting memory and hereditary properties.

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