

SOME GENERALIZATIONS OF FRACTIONAL INTEGRAL INEQUALITIES

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Abstract. In this paper we generalize some fractional integral inequalities proved by Z. Dahmani and al. (see [2]).

Index Terms. Inequalities, fractional integral.

I. INTRODUCTION

The Chebyshev fractional for two integrable functions f and g which are synchronous ($i, e (f(x) - f(y))(g(x) - g(y)) \geq 0$) for any $x, y \in [a, b]$, is defined as follows

$$T(f, g) = \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \frac{1}{b-a} \int_a^b g(x)dx. \quad (1)$$

The functional has attracted many researchers attention due to applications in mathematics and statistics and a number of inequalities associated with it. (See [3], [4] and [5]).

We recall some definitions relating to the fractional integration in the sense of Riemann-Liouville.

Definition 1 Let $f \in L_1([0, \infty))$. The Riemann-Liouville fractional integral of order $\alpha \geq 0$ is defined by

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(x)}{(t-x)^{\alpha-1}} dx, \alpha > 0, t > a,$$

$$I^0 f(t) = f(t), \quad (2)$$

where $\Gamma(\alpha) = \int_0^\infty \exp(-t) t^{\alpha-1} dt$.

When $f(x) = x^\mu$ we get the another expression that will be used later

$$I^\alpha x^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} x^{\mu+\alpha}, \quad \alpha > 0, \mu > -1, x > 0. \quad (3)$$

Remark 1 In the case where f is defined on the space $L_1([a, b])$, $a < b$, we have the following definition.

Definition 2 The Riemann-Liouville fractional operator of order $\alpha \geq 0$ for a function $f \in L_1([a, b])$, is defined by

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(x)}{(t-x)^{1-\alpha}} dx, \alpha > 0, a < t \leq b,$$

$$I^0 f(t) = f(t). \quad (4)$$

Definition 3 Let $f \in L_1([a, b])$, $a < b$, $\alpha > 0$, $k > 0$. The operator $I_{0,k}^\alpha$ is defined by

$$I_{0,k}^\alpha f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_0^t (t-x)^{\frac{\alpha}{k}-1} f(x) dx, \quad (5)$$

where

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} \exp\left(-\frac{t^k}{k}\right) dt, k > 0.$$

II. MAIN RESULTS

Lemma 1 Let $f \in L_1([0, \infty))$ and $t > 0$, $\alpha > 0$, $k > 0$. Then

$$I_{0,k}^\alpha(f)(t) = \frac{1}{k\Gamma_k(\alpha)} I_k^\alpha f(t) \Gamma\left(\frac{\alpha}{k}\right). \quad (6)$$

Proof. For all $f \in L_1([0, \infty))$ and $t > 0$, $\alpha > 0$, $k > 0$, we have

$$I_k^\alpha f(t) = \frac{1}{\Gamma\left(\frac{\alpha}{k}\right)} \int_0^t (t-x)^{\frac{\alpha}{k}-1} f(x) dx,$$

and

$$I_{0,k}^\alpha f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_0^t (t-x)^{\frac{\alpha}{k}-1} f(x) dx.$$

Then

$$\int_0^t (t-x)^{\frac{\alpha}{k}-1} f(x) dx = I_k^\alpha f(t) \Gamma\left(\frac{\alpha}{k}\right),$$

finally

$$I_{0,k}^\alpha(f)(t) k\Gamma_k(\alpha) = I_k^\alpha f(t) \Gamma\left(\frac{\alpha}{k}\right).$$

Lemma 1 is proved.

Theorem 1 Let the functions f and g be two synchronous functions on $[0, \infty]$, then for all $t > 0$, $\alpha > 0$, $k > 0$

$$I_{0,k}^\alpha(fg)(t) \geq \frac{1}{I_{0,k}^\alpha(1)} I_{0,k}^\alpha(f)(t) I_{0,k}^\alpha(g)(t). \quad (7)$$

Where $I_{0,k}^\alpha(1) = \frac{t^{\frac{\alpha}{k}}}{\alpha\Gamma_k(\alpha)}$.

Proof. The functions f and g are synchronous functions on $[0, \infty[$, then for all $\tau \geq 0, \rho \geq 0$, we get

$$(f(\tau) - f(\rho))(g(\tau) - g(\rho)) \geq 0 \quad (8)$$

and

$$f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau). \quad (9)$$

Multiplying both hand sides of (9) by $\frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}$, $\tau \in (0, t)$, we get

$$\begin{aligned} & \frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} f(\tau)g(\tau) + \frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} f(\rho)g(\rho) \\ & \geq \frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} f(\tau)g(\rho) + \frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} f(\rho)g(\tau). \end{aligned} \quad (10)$$

Integrating (10) over $(0, t)$, we obtain

$$\begin{aligned} & \frac{1}{k\Gamma_k(\alpha)} \int_0^t (t-\tau)^{\frac{\alpha}{k}-1} f(\tau)g(\tau) d\tau \\ & + \frac{1}{k\Gamma_k(\alpha)} \int_0^t (t-\tau)^{\frac{\alpha}{k}-1} f(\rho)g(\rho) d\tau \\ & \geq \frac{1}{k\Gamma_k(\alpha)} \int_0^t (t-\tau)^{\frac{\alpha}{k}-1} f(\tau)g(\rho) d\tau \\ & + \frac{1}{k\Gamma_k(\alpha)} \int_0^t (t-\tau)^{\frac{\alpha}{k}-1} f(\rho)g(\tau) d\tau. \end{aligned} \quad (11)$$

Consequently

$$\begin{aligned} & I_{0,k}^\alpha(fg)(t) + f(\rho)g(\rho) \frac{1}{k\Gamma_k(\alpha)} \int_0^t (t-\tau)^{\frac{\alpha}{k}-1} d\tau \\ & \geq \frac{g(\rho)}{k\Gamma_k(\alpha)} \int_0^t (t-\tau)^{\frac{\alpha}{k}-1} f(\tau) d\tau \\ & + \frac{f(\rho)}{k\Gamma_k(\alpha)} \int_0^t (t-\tau)^{\frac{\alpha}{k}-1} g(\tau) d\tau. \end{aligned} \quad (12)$$

So we get

$$\begin{aligned} & I_{0,k}^\alpha(fg)(t) + f(\rho)g(\rho) I_{0,k}^\alpha(1) \\ & \geq g(\rho) I_{0,k}^\alpha(f)(t) + f(\rho) I_{0,k}^\alpha(g)(t). \end{aligned} \quad (13)$$

Now multiplying both hand sides of (13) by $\frac{(t-\rho)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}$, $\rho \in (0, t)$, we obtain

$$\begin{aligned} & \frac{(t-\rho)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} I_{0,k}^\alpha(fg)(t) + \frac{(t-\rho)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} f(\rho)g(\rho) I_{0,k}^\alpha(1) \geq \\ & \frac{(t-\rho)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} g(\rho) I_{0,k}^\alpha(f)(t) + \\ & \frac{(t-\rho)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} f(\rho) I_{0,k}^\alpha(g)(t). \end{aligned} \quad (14)$$

By integrating (14) over $(0, t)$, it follows

$$\begin{aligned} & I_{0,k}^\alpha(fg)(t) \int_0^t \frac{(t-\rho)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} d\rho \\ & + \frac{I_{0,k}^\alpha(1)}{k\Gamma_k(\alpha)} \int_0^t f(\rho)g(\rho) (t-\rho)^{\frac{\alpha}{k}-1} d\rho \\ & \geq \frac{I_{0,k}^\alpha(f)(t)}{k\Gamma_k(\alpha)} \int_0^t (t-\rho)^{\frac{\alpha}{k}-1} g(\rho) d\rho \\ & + \frac{I_{0,k}^\alpha(g)(t)}{k\Gamma_k(\alpha)} \int_0^t (t-\rho)^{\frac{\alpha}{k}-1} f(\rho) d\rho. \end{aligned} \quad (15)$$

Therefore

$$I_{0,k}^\alpha(fg)(t) \geq \frac{1}{I_{0,k}^\alpha(1)} I_{0,k}^\alpha(f)(t) I_{0,k}^\alpha(g)(t) \quad (16)$$

The proof is complete.

Similarly one can prove the following corollary.

Corollary 1 If the functions f and g are asynchronous (i.e. $(f(\tau) - f(\rho))(g(\tau) - g(\rho)) \leq 0$), for any $\tau, \rho \in [a, b]$, then

$$I_{0,k}^\alpha(fg)(t) \leq \frac{1}{I_{0,k}^\alpha(1)} I_{0,k}^\alpha(f)(t) I_{0,k}^\alpha(g)(t) \quad (17)$$

Theorem 2 Let f and g be two synchronous functions on $[0, \infty[$. Then for all $t > 0, k > 0, \alpha > 0, \beta > 0$, the following inequality

$$\begin{aligned} & \frac{t^{\frac{\alpha}{k}}}{\alpha\Gamma_k(\alpha)} I_{0,k}^\beta(fg)(t) + \frac{t^{\frac{\beta}{k}}}{\beta\Gamma_k(\beta)} I_{0,k}^\alpha(fg)(t) \\ & \geq I_{0,k}^\alpha(f)(t) I_{0,k}^\beta(g)(t) + I_{0,k}^\beta(f)(t) I_{0,k}^\alpha(g)(t), \end{aligned} \quad (18)$$

holds.

Proof. Applying the proof of Theorem 1, we get

$$\begin{aligned} & \frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)} I_{0,k}^\alpha(fg)(t) + I_{0,k}^\alpha(1) \frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)} f(\rho)g(\rho) \\ & \geq \frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)} g(\rho) I_{0,k}^\alpha(f)(t) + \frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)} f(\rho) I_{0,k}^\alpha(g)(t). \end{aligned} \quad (19)$$

Integrating (19) over $(0, t)$, we obtain

$$\begin{aligned} & I_{0,k}^\alpha(fg)(t) \int_0^t \frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)} d\rho \\ & + \frac{I_{0,k}^\alpha(1)}{k\Gamma_k(\beta)} \int_0^t f(\rho)g(\rho)(t-\rho)^{\frac{\beta}{k}-1} d\rho \\ & \geq \frac{I_{0,k}^\alpha(f)(t)}{k\Gamma_k(\beta)} \int_0^t (t-\rho)^{\frac{\beta}{k}-1} g(\rho) d\rho \\ & + \frac{I_{0,k}^\alpha(g)(t)}{k\Gamma_k(\beta)} \int_0^t (t-\rho)^{\frac{\beta}{k}-1} f(\rho) d\rho. \end{aligned} \quad (20)$$

The proof is complete.

Corollary 2 If the functions f and g are asynchronous, then inequality (20) holds in the reversed direction.

Remark 1 If $\alpha = \beta$ in Theorem 2, we obtain inequality (7) of Theorem 1.

Theorem 3 Let $(f_i)_{i=1,\dots,n}$ be n positive increasing functions on $[0, \infty]$. Then for any $t > 0$, $\alpha > 0$, $k > 0$, we have

$$I_{0,k}^\alpha(\Pi_{i=1}^n f_i)(t) \geq I_{0,k}^\alpha(1)^{1-n} \Pi_{i=1}^n I_{0,k}^\alpha f_i(t). \quad (21)$$

Proof. We proceed via mathematical induction. For $n = 2$, applying (7), we obtain

$$I_{0,k}^\alpha(f_1 f_2)(t) \geq (I_{0,k}^\alpha(1))^{-1} I_{0,k}^\alpha f_1(t) I_{0,k}^\alpha f_2(t),$$

for all $t > 0, \alpha > 0, k > 0$.

Now, suppose that

$$I_{0,k}^\alpha(\Pi_{i=1}^{n-1} f_i)(t) \geq (I_{0,k}^\alpha(1))^{2-n} \Pi_{i=1}^{n-1} I_{0,k}^\alpha f_i(t) \quad (22)$$

for all $t > 0, \alpha > 0, k > 0$.

Since $(f_i)_{i=1,\dots,n}$ are positive increasing functions, then $(\Pi_{i=1}^{n-1} f_i)(t)$ is an increasing function, then we can apply Theorem 1 to the functions $\Pi_{i=1}^{n-1} f_i = g, f_n = f$. Consequently

$$I_{0,k}^\alpha(\Pi_{i=1}^n f_i)(t) = I_{0,k}^\alpha(fg)(t)$$

$$\geq (I_{0,k}^\alpha(1))^{-1} I_{0,k}^\alpha(\Pi_{i=1}^{n-1} f_i)(t) I_{0,k}^\alpha f_n(t).$$

From hypothesis (22), we deduce

$$\begin{aligned} & I_{0,k}^\alpha(\Pi_{i=1}^n f_i)(t) \\ & \geq (I_{0,k}^\alpha(1))^{-1} (I_{0,k}^\alpha(1))^{2-n} \left(I_{0,k}^\alpha(\Pi_{i=1}^{n-1} f_i)(t) \right) I_{0,k}^\alpha(f_n)(t). \end{aligned} \quad (23)$$

The proof is complete.

Theorem 4 Let f and g be two functions defined on $[0, \infty]$, such that f is increasing and g is differentiable and there is a real number $m = \inf_{t>0} g'(t)$. Then the following inequality

$$\begin{aligned} & I_{0,k}^\alpha(fg)(t) \geq I_{0,k}^\alpha(1)^{-1} I_{0,k}^\alpha f(t) I_{0,k}^\alpha g(t) \\ & - \frac{mt}{\frac{\alpha}{k} + 1} I_{0,k}^\alpha f(t) + m I_{0,k}^\alpha(t f(t)). \end{aligned} \quad (24)$$

Holds for all $t > 0, \alpha > 0$ and $k > 0$.

Proof. Let $h(t) := g(t) - mt$. It is obviously that h is differentiable and increasing on $[0, \infty]$. Then by applying Theorem 1 and inequality (3), we find

$$\begin{aligned} & I_{0,k}^\alpha((g - mt)f(t)) \\ & \geq (I_{0,k}^\alpha(1))^{-1} I_{0,k}^\alpha f(t) (I_{0,k}^\alpha g(t) - m I_{0,k}^\alpha(t)) \\ & \geq (I_{0,k}^\alpha(1))^{-1} I_{0,k}^\alpha f(t) I_{0,k}^\alpha g(t) - m (I_{0,k}^\alpha(1))^{-1} I_{0,k}^\alpha(t) I_{0,k}^\alpha f(t) \\ & \geq (I_{0,k}^\alpha(1))^{-1} I_{0,k}^\alpha f(t) I_{0,k}^\alpha g(t) - m (I_{0,k}^\alpha(1))^{-1} I_{0,k}^\alpha(t) I_{0,k}^\alpha f(t) \\ & \geq (I_{0,k}^\alpha(1))^{-1} I_{0,k}^\alpha f(t) I_{0,k}^\alpha g(t) - \frac{m(I_{0,k}^\alpha(1))^{-1} t^{\frac{\alpha}{k}+1}}{\Gamma(\frac{\alpha}{k} + 2)} I_{0,k}^\alpha f(t) \\ & \geq (I_{0,k}^\alpha(1))^{-1} I_{0,k}^\alpha f(t) I_{0,k}^\alpha g(t) - \frac{m \Gamma(\frac{\alpha}{k} + 1) t}{\Gamma(\frac{\alpha}{k} + 2)} I_{0,k}^\alpha f(t) \\ & \geq (I_{0,k}^\alpha(1))^{-1} I_{0,k}^\alpha f(t) I_{0,k}^\alpha g(t) - \frac{mt}{\frac{\alpha}{k} + 1} I_{0,k}^\alpha f(t). \end{aligned}$$

Hence

$$\begin{aligned} & I_{0,k}^\alpha(fg)(t) \geq (I_{0,k}^\alpha(1))^{-1} I_{0,k}^\alpha f(t) I_{0,k}^\alpha g(t) - \frac{mt}{\frac{\alpha}{k} + 1} I_{0,k}^\alpha f(t) \\ & + m I_{0,k}^\alpha(t f(t)), \quad t > 0, \alpha > 0, k > 0. \end{aligned}$$

The proof is complete.

Corollary 3 Let f and g be two functions defined on $[0, \infty]$.

a) Suppose that f is decreasing, g is differentiable and there is a real number $M := \sup_{t \geq 0} g'(t)$, then for all $t > 0, \alpha > 0, k > 0$, then the following inequality

$$I_{0,k}^\alpha(fg)(t) \geq (I_{0,k}^\alpha(1))^{-1} I_{0,k}^\alpha f(t) I_{0,k}^\alpha g(t) - \frac{Mt}{\frac{\alpha}{k} + 1} I_{0,k}^\alpha f(t) + M I_{0,k}^\alpha(t f(t)). \quad (26)$$

Holds.

b) Suppose that f and g are differentiable and there is $m_1 := \inf_{t \geq 0} f'(t), m_2 := \inf_{t \geq 0} g'(t)$, then the following inequality

$$I_{0,k}^\alpha(fg)(t) - m_1 I_{0,k}^\alpha t g(t) - m_2 I_{0,k}^\alpha t f(t) + m_1 m_2 I_{0,k}^\alpha t^2 \geq (I_{0,k}^\alpha(1))^{-1} \left(I_{0,k}^\alpha f(t) I_{0,k}^\alpha g(t) - m_1 I_{0,k}^\alpha(t) I_{0,k}^\alpha g(t) - m_2 I_{0,k}^\alpha(t) I_{0,k}^\alpha f(t) + m_1 m_2 (I_{0,k}^\alpha(t))^2 \right). \quad (26)$$

c) Suppose that f and g are differentiable and there is $M_1 := \sup_{t \geq 0} f'(t), M_2 := \sup_{t \geq 0} g'(t)$, then the following inequality

$$I_{0,k}^\alpha(fg)(t) - M_1 I_{0,k}^\alpha t g(t) - M_2 I_{0,k}^\alpha t f(t) + M_1 M_2 I_{0,k}^\alpha t^2 \geq (I_{0,k}^\alpha(1))^{-1} \left(I_{0,k}^\alpha f(t) I_{0,k}^\alpha g(t) - M_1 I_{0,k}^\alpha(t) I_{0,k}^\alpha g(t) - M_2 I_{0,k}^\alpha(t) I_{0,k}^\alpha f(t) + M_1 M_2 (I_{0,k}^\alpha(t))^2 \right). \quad (27)$$

holds.

Proof.

a) By Applying Theorem 1 to the decreasing functions f and G such that

$$G(t) := g(t) - Mt,$$

we get (25).

b) Using Theorem 1 to the increasing functions F and G where

$$F(t) := f(t) - m_1 t, \quad G(t) := g(t) - m_2 t,$$

we find that

$$I_{0,k}^\alpha((g - m_2 t)(f - m_1 t)) \geq (I_{0,k}^\alpha(1))^{-1} [(I_{0,k}^\alpha g - m_2 I_{0,k}^\alpha t)(I_{0,k}^\alpha f - m_1 I_{0,k}^\alpha t)] = (I_{0,k}^\alpha(1))^{-1} [I_{0,k}^\alpha f I_{0,k}^\alpha g - m_1 I_{0,k}^\alpha t I_{0,k}^\alpha g - m_2 I_{0,k}^\alpha t I_{0,k}^\alpha f + m_1 m_2 (I_{0,k}^\alpha t)^2],$$

where

$$I_{0,k}^\alpha((g - m_2 t)(f - m_1 t)) =$$

$$I_{0,k}^\alpha f g(t) - m_1 I_{0,k}^\alpha(t g(t)) - m_2 I_{0,k}^\alpha(t f(t)) + m_1 m_2 I_{0,k}^\alpha t^2.$$

To prove c), it suffice to consider two decreasing functions

$$F(t) := f(t) - M_1 t, \quad G(t) := g(t) - M_2 t.$$

Threfore

$$I_{0,k}^\alpha((g - M_2 t)(f - M_1 t)) \geq$$

$$(I_{0,k}^\alpha(1))^{-1} [(I_{0,k}^\alpha g - M_2 I_{0,k}^\alpha t)(I_{0,k}^\alpha f - M_1 I_{0,k}^\alpha t)] =$$

$$(I_{0,k}^\alpha(1))^{-1} [I_{0,k}^\alpha f I_{0,k}^\alpha g - M_1 I_{0,k}^\alpha t I_{0,k}^\alpha g - M_2 I_{0,k}^\alpha t I_{0,k}^\alpha f + M_1 M_2 (I_{0,k}^\alpha t)^2].$$

where

$$I_{0,k}^\alpha((g - M_2 t)(f - M_1 t)) =$$

$$I_{0,k}^\alpha f g(t) - M_1 I_{0,k}^\alpha(t g(t)) - M_2 I_{0,k}^\alpha(t f(t)) + M_1 M_2 I_{0,k}^\alpha t^2.$$

The proof of Corollary3 is complete.

Remark. If $k = 1$ in Theorems 1, 2, 3, 4 and Corollaries, we obtain the results of Theorems 3.1, 3.2, 3.5, 3.6 and Corollary 3.7 of [2] respectively.

III. APPLICATIONS

We consider some applications of notion of synchronous functions.

1. First we give a new proof of Theorem1 for affine functions $f(t) = a_1 t + b_1$ and $g(t) = a_2 t + b_2$ under condition $a_1 a_2 > 0$ which are a particular case of synchronous functions.

Corollary 4. Let the functions f and g be two affine functions $f(t) = a_1 t + b_1$ and $g(t) = a_2 t + b_2$ on $[0, \infty[$, where $a_1 a_2 > 0, t > 0, \alpha > 0, k > 0$, then

$$I_{0,k}^\alpha(a_1 t + b_1)(a_2 t + b_2) \geq \frac{1}{I_{0,k}^\alpha(1)} I_{0,k}^\alpha(a_1 t + b_1) I_{0,k}^\alpha(a_2 t + b_2). \quad (28)$$

Proof.

The following inequality

$$2\left(\frac{\alpha}{k} + 1\right) \geq \frac{\alpha}{k} + 2, \quad (29)$$

holds for $\alpha, k > 0$.

Inequality (29) implies

$$\frac{2}{\frac{\alpha}{k} + 2} \geq \frac{1}{\frac{\alpha}{k} + 1}. \quad (30)$$

Multiplying both hand sides of (30) by

$$a_1 a_2 \frac{1}{k \Gamma_k(\alpha)} \Gamma\left(\frac{\alpha}{k}\right) \frac{\Gamma(2)}{\Gamma\left(2 + \frac{\alpha}{k}\right)} t^{2 + \frac{\alpha}{k}}, \text{ where}$$

$a_1 a_2 > 0, t > 0$, we get

$$\frac{2}{\frac{\alpha}{k} + 2} a_1 a_2 \frac{1}{k \Gamma_k(\alpha)} \Gamma\left(\frac{\alpha}{k}\right) \frac{\Gamma(2)}{\Gamma\left(2 + \frac{\alpha}{k}\right)} t^{2 + \frac{\alpha}{k}} \geq$$

$$\frac{1}{\frac{\alpha}{k}+1} a_1 a_2 \frac{1}{k\Gamma_k(\alpha)} \Gamma\left(\frac{\alpha}{k}\right) \frac{\Gamma(2)}{\Gamma(2+\frac{\alpha}{k})} t^{2+\frac{\alpha}{k}}.$$

Since

$$\Gamma(z+1) = z\Gamma(z), \quad (31)$$

for all $z > 0$, then by (31), we get

$$a_1 a_2 \frac{1}{k\Gamma_k(\alpha)} \Gamma\left(\frac{\alpha}{k}\right) \frac{\Gamma(3)}{\Gamma(3+\frac{\alpha}{k})} t^{2+\frac{\alpha}{k}} \geq a_1 a_2 \frac{t}{\frac{\alpha}{k}+1} \frac{1}{k\Gamma_k(\alpha)} \Gamma\left(\frac{\alpha}{k}\right) \frac{\Gamma(2)}{\Gamma(2+\frac{\alpha}{k})} t^{1+\frac{\alpha}{k}}. \quad (32)$$

Using lemma 1 and equality (3) respectively, we obtain

$$\begin{aligned} I_{0,k}^\alpha(1) &= \frac{1}{k\Gamma_k(\alpha)} \Gamma\left(\frac{\alpha}{k}\right) I_k^\alpha(1) \\ &= \frac{1}{k\Gamma_k(\alpha)} \Gamma\left(\frac{\alpha}{k}\right) \frac{\Gamma(1)}{\Gamma(1+\frac{\alpha}{k})} t^{\frac{\alpha}{k}}. \end{aligned} \quad (33)$$

Similarly, we have

$$\begin{aligned} I_{0,k}^\alpha(t) &= \frac{1}{k\Gamma_k(\alpha)} \Gamma\left(\frac{\alpha}{k}\right) I_k^\alpha(t) \\ &= \frac{1}{k\Gamma_k(\alpha)} \Gamma\left(\frac{\alpha}{k}\right) \frac{\Gamma(2)}{\Gamma(2+\frac{\alpha}{k})} t^{\frac{\alpha}{k}+1}, \end{aligned} \quad (34)$$

and

$$\begin{aligned} I_{0,k}^\alpha(t^2) &= \frac{1}{k\Gamma_k(\alpha)} \Gamma\left(\frac{\alpha}{k}\right) I_k^\alpha(t^2) \\ &= \frac{1}{k\Gamma_k(\alpha)} \Gamma\left(\frac{\alpha}{k}\right) \frac{\Gamma(3)}{\Gamma(3+\frac{\alpha}{k})} t^{\frac{\alpha}{k}+2}. \end{aligned} \quad (35)$$

Thus

$$\frac{I_{0,k}^\alpha(t)}{I_{0,k}^\alpha(1)} = \frac{\frac{\Gamma(2)}{\Gamma(2+\frac{\alpha}{k})} t^{\frac{\alpha}{k}+1}}{\frac{\Gamma(1)}{\Gamma(1+\frac{\alpha}{k})} t^{\frac{\alpha}{k}}} = \frac{\Gamma(1+\frac{\alpha}{k})}{\Gamma(2+\frac{\alpha}{k})} t = \frac{t}{1+\frac{\alpha}{k}}. \quad (36)$$

By applying (34),(35) and (36), we find

$$a_1 a_2 I_{0,k}^\alpha(t^2) \geq a_1 a_2 \frac{1}{I_{0,k}^\alpha(1)} (I_{0,k}^\alpha(t))^2. \quad (37)$$

By adding $(a_1 b_2 + a_2 b_1) I_{0,k}^\alpha(t) + b_1 b_2 I_{0,k}^\alpha(1)$ to both hand sides of (37), we get

$$\begin{aligned} a_1 a_2 I_{0,k}^\alpha(t^2) + (a_1 b_2 + a_2 b_1) I_{0,k}^\alpha(t) + b_1 b_2 I_{0,k}^\alpha(1) &\geq \\ a_1 a_2 \frac{1}{I_{0,k}^\alpha(1)} (I_{0,k}^\alpha(t))^2 + (a_1 b_2 + a_2 b_1) I_{0,k}^\alpha(t) &+ \\ + b_1 b_2 I_{0,k}^\alpha(1), \end{aligned} \quad (38)$$

where $b_1, b_2 \in \mathbb{R}$.

Consequently

$$\begin{aligned} I_{0,k}^\alpha(a_1 a_2 t^2 + (a_1 b_2 + a_2 b_1)t + b_1 b_2) &\geq \\ \frac{1}{I_{0,k}^\alpha(1)} \left[(a_1 a_2 (I_{0,k}^\alpha(t))^2 + (a_1 b_2 + a_2 b_1) I_{0,k}^\alpha(1) I_{0,k}^\alpha(t) + \right. & \\ \left. (b_1 b_2 I_{0,k}^\alpha(1))^2 \right]. \end{aligned} \quad (39)$$

Therefore

$$\begin{aligned} I_{0,k}^\alpha(a_1 t + b_1)(a_2 t + b_2) &\geq \\ \frac{1}{I_{0,k}^\alpha(1)} I_{0,k}^\alpha(a_1 t + b_1) I_{0,k}^\alpha(a_2 t + b_2). \end{aligned}$$

Inequality (28) is particular case of Theorem1 with

$$f(t) = a_1 t + b_1 \text{ and } g(t) = a_2 t + b_2.$$

2. Second, the notion of synchronous functions admits certain applications in computer science, for example in the theory of parallelism.

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References

- [1] P. L. Chebyshev, Sur les expressions approximatives des intégrales définies par les autres prises entre les mêmes limites, Proc. Math. Soc. Charkov, 2 (1882), 93-98.
- [2] S. Belarbi and Z. Dahmani, On some new fractional integral inequalities. Int. Journal of Math. Analysis, vol 4 2010, no. 4, 185-191.
- [3] P. Cerone and S.S. Dargomir, Some new Ostrowski-type bounds for the Chebyshev functional and applications, J. Math. Inequal., 8 (2014), 159-170.
- [4] G. Rahman, Z. Ullah, A. Khan, E. Set, Korakkaran Sooppy Nisar, Certain Chybeshev-type inequalities. Involving fractional conformable integral operators, Mathematics 2019, 7, 364, doi :0.3390, math 7040364.
- [5] F. Gorenflo and F. Maindari, Fractional calculus: integral and differential equations fractional order, Springer Verlag, Wien, 1997, pp. 223-276.