Theoretical and numerical results for the 2-d of the regularised saint-venant system coupled with the transport equation

Mohammed Lachache, Fatma Zohra Nouri

LAM2SIN, Faculty of Sciences, Badji Mokhtar University, P.O. Box 12, 23005, Annaba, Algeria

Article Info

Article history:

Received: 21/01/2025
Revised: 26/04/2025
Accepted: 30/04/2025

Keywords:

Saint-Venant System,
Well-Posedness,
Numerical Simulation

ABSTRACT

In this work, we study the transport of a passive pollutant by a flow modeled using the Saint-Venant system. For that, we take into ac- count a regularisation of the Saint-Venant system to describe the flows in rivers and lakes, to which we add a transport equation. We derive a mathematical model and present theoretical results on its well-posedness. Finally, using a reliable finite difference scheme, we discretized the proposed model and provide numerical results to demonstrate the effective behavior of the phenomena

Corresponding Author: mohamedlachache1996@gmail.com

1. INTRODUCTION

The Saint-Venant system describes the flows in river and lake hydrodynamics. The first version of the system was developed by Saint-Venant in 1871 [10], while another version was derived from the Navier-Stokes equations by Gerbeau et al. in 2000 [7]. Over the past two decades, many authors have shown interest in these types of equations. For example, see [3], [6], [8], and [9]. Numerous numerical methods have been proposed for simulating surface flow in the Saint- Venant system, such as finite difference, element, and volume methods. See, for example, [1], [8] and [9] ...etc.

However, the transfer of pollution in flows remains a hazard both environmentally and industrially. Several mathematical models have been presented by authors to study the propagation of pollutants. For instance, refer to [4] and [13]. The coupling of the Saint-Venant system with advection-diffusion type equations has been explored by many authors to compute the transport of passive pollutants using the Saint-Venant system. See, for example, [5], [11], [12] and [14]. The objective of this article is to examine a model that regularizes the Saint-Venant system coupled with a transport equation for pollutant movement. The primary goal is to establish theoretical results regarding the well-posedness of the model. Subsequently, we explore the discretizations of the proposed model using a stable finite difference scheme in both temporal and spatial domains.

2. MODEL DESCRIPTION

Let $\Omega = (0, T) \times I$ be a domain, such that I is a closed bounded domain in $\mathbb{R}^2$ and T > 0, we consider the following coupled system

$$\begin{cases} \dfrac{\partial h}{\partial t} + div(hU) = S, \\ \dfrac{\partial hU}{\partial t} + div(hU \otimes U) + \nabla(\dfrac{g}{2}h^2) + \nabla(\varepsilon R h^2) + gh\nabla Z = 0, \\ \dfrac{\partial hc}{\partial t} + div(hUc) = c_S S, \\ h(0,x,y) = h_0(x,y), \ \ \forall (x,y) \in I, \\ U(0,x,y) = U_0(x,y), \ \ \forall (x,y) \in I, \\ c(0,x,y) = c_0(x,y), \ \ \forall (x,y) \in I, \end{cases} \tag{1}$$

where $R = (R1, R2)$) such that

©UBMA – 2025

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} h((\frac{\partial u}{\partial x})^2 - \frac{\partial u_t}{\partial x} - u\frac{\partial^2 u}{\partial x^2}) - g(h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}(\frac{\partial h}{\partial x})^2) \\ h((\frac{\partial v}{\partial y})^2 - \frac{\partial v_t}{\partial y} - v\frac{\partial^2 v}{\partial y^2}) - g(h\frac{\partial^2 h}{\partial y^2} + \frac{1}{2}(\frac{\partial h}{\partial y})^2) \end{pmatrix}, \quad (2)$$

and ε is a regularisation parameter $(0 < \varepsilon < 1)$.
The first two equations of (1) represent a regularization version of Saint-Venant system while the third one is the classical transport equation. We can write this system in the conservative and compact form

$$\frac{\partial \varphi}{\partial t} + divF(\varphi) = B(\varphi), \quad (3)$$

$$\text{with } \varphi = \begin{pmatrix} h \\ q \\ e \end{pmatrix}, F(\varphi) = \begin{pmatrix} q \\ \frac{q\otimes q}{h} + \frac{gh^2}{2}Id + \varepsilon Rh^2 \\ \frac{qe}{h} \end{pmatrix}, B(\varphi) = \begin{pmatrix} S \\ -gh\nabla Z \\ T_s S \end{pmatrix},$$

such that the flux of water is $q\,(t,x,y) = h\,(t,x,y)\,U(t,x,y)$ and the quantity of pollutant is denoted by $e\,(t,x,y) = h\,(t,x,y)\,c\,(t,x,y)$.

3. MAIN RESULTS

Here, we present a kinetic approach to (3) in homogeneous case, i.e $S = 0$. Let $\chi(w)$ be an even and compactly supported probability Then we introduce two microscopic densities of particles $M\,(t,x,\xi)$ and $N\,(t,x,\xi)$ defined by a Gibbs equilibrium

$$M(t,x,y,\xi) = \frac{h(t,x,y)}{c(t,x,y)}\chi\left(\frac{\xi - U(x,y)}{c(t,x,y)}\right),$$

$$N(t,x,y,\xi) = \frac{e(t,x,y)}{c(t,x,y)}\chi\left(\frac{\xi - U(x,y)}{c(t,x,y)}\right),$$

$$c(t,x,y)^2 = h(t,x,y)\left(\frac{g}{2}Id + \varepsilon R(t,x,y)\right).$$

We denote by

$$G(t,x,y,\xi) = \begin{pmatrix} M(t,x,y,\xi) \\ N(t,x,y,\xi) \end{pmatrix}.$$

we use a kinetic approach to prove the well posedness of (3), by presenting the following theorem

Theorem 1 *The functions* (h, q, e) *are weak solution of* (3) *if and only if* $G\,(t, x, y, \xi)$ *is solution of the kinetic equation*

$$\frac{\partial G}{\partial t} + \xi \cdot \nabla G - g\nabla Z \nabla_\xi G = Q(t,x,y,\xi), \quad (4)$$

where, $Q(t,x,y,\xi) = \begin{pmatrix} Q_1(t,x,y,\xi) \\ Q_2(t,x,y,\xi) \end{pmatrix}$, *is a "collision term" satisfying*

$$\int_{R^2} \begin{pmatrix} 1 \\ \xi \end{pmatrix} Q_1 d\xi = 0, \quad \int_{R^2} Q_2 d\xi = 0. \quad (5)$$

©UBMA – 2025

proof. System (3) are equivalent to (4) once integrated in ξ against $K(\xi)$ such that

$$K(\xi) = \begin{pmatrix} 1 & 0 \\ \xi & 0 \\ 0 & 1 \end{pmatrix}. \tag{6}$$

Refer to [3] for a full verification of the hydrodynamic element of the kinetic interpretation, and [14] for the handling of the source term at this microscopic level

4. NUMERICAL RESULTS

System (1) is approximated using an explicit finite difference method, and we provide a numerical stability finding of the schemes by using Fourier analysis written the solution in the following form

$$h_{I,j}^n = \lambda^n e^{i\pi(kI+mj)}, u_{I,j}^n = \lambda^n e^{i\pi(kI+mj)}, c_{I,j}^n = \lambda^n e^{i\pi(kI+mj)}, \tag{7}$$

Comments on the results

First numerical experimentation

The obtained results are resumed in Figures $1-5$, showing the variation, at $t = 2$ & 3, of the flow height and the propagation of pollution in the flow for the SV-T solution (on the left of the Figures) and the RSV-T solution (on the right of the Figures), for $\varepsilon = 10^{-3}$.
We see that the changes are exactly in the wave curvature, which is clearly seen in the flow height and flow velocity in Figures 4 & 5 at time $t = 3$, where slight variances in the pollution propagation are noticed, when the wave curvature in the RSV-T is larger. Note that the results remain stable.

Second numerical experimentation

If we take epsilon two times larger ($\varepsilon = 2 \cdot 10^{-3}$), the stability is maintained with an increase of the wave curvature in magnitude of *h*, *v*, *u* and *c* solutions of (1), see Figures 6.

5. CONCLUSION

In this work, we studied the transport of a passive pollutant in a flow, using the Saint-Venant system. In fact we considered a regularisation version of Saint-Venant system coupled with a transport equation. We first proved theoretical results on its well-posedness, then we focused on the numerical study of the related descretised system. In order to understand the flow height behavior and the pollutant propagation in the flow at different time and space locations, we proposed a numerical method, based on a simple explicit finite difference scheme, we derived stability conditions, so called CFL conditions. Different cases were studied, where the obtained results are resumed in Figures (1-6). We deduct from further numerical experimentation that when Ɛ>2.10-3, instability occurs.

6. REFERENCES

[1] H. Altaie, *Numerical Solutions for 2D Depth-Averaged Shallow Water Equations*, International Mathematical Forum, Vol. 13, 2018, no. 2, 79 - 90.

[2] E. Audusse, M.O. Bristeau, Transport of pollutant in shallow water a two time steps kinetic method, Vol. 37, No 2, 2003, pp. 389–416.

[3] D. BRESCH, B. DESJARDINS. Sur un modèle de Saint-Venant visqueux et sa limite quasigéostrophique. C.R. Acad. Paris, Ser. I 335 (2002), 973–978.

[4] L. Cai, W. Xie, J. Feng, J. Zhou. Computations of transport of pollutant in shallow water. Applied Mathematical Modelling 31, 490–498, 2007.

[5] A. Chertock, A. Kurganov, G. Petrova, Finite-Volume-Particle Methods for Models of Transport of Pollutant in Shallow Water, Journal of Scientific Computing, Vol. 27, Nos. 1–3, June 2006.

[6] D. Dutykh and D. Mitsotakis. On the relevance of the dam break problem in the context of nonlinear shallow water equations. Discrete and Continuous Dynamical Systems - Series B, 13(4) :799–818, 2010.

[7] F. GERBEAU, B. P ERTHAME. Derivation of viscous Saint-Venant system for laminar shallow water; Numerical results. Discrete and Continuous Dynamical Systems-series B. 1, No. 1, (2001), 89–102.

[8] S.A. Kamboh, N. S, L. Izzatul, O.E. Monday, *Simulation of 2D Saint-Venant equations in open channel by using MATLAB; Journal of IT in Asia*, Vol 5, (2015).

[9] Sy. Mamadou, K. Souleye, B. Olivier, *On a 1D-Shallow Water Model : Existence of solution and numerical simulations*, Revue Africaine de la Recherche en Informatique et Mathématiques Appliquées, volume 9 (2008)

[10] A.J.C. Saint-Venant. Théorie du mouvement non-permanent des eaux, avec application aux crues des rivières et à l'introduction des marées dans leur lit. C. R. Acad. Sc. Paris, 73 :147–154,1871.

[11] R. Touma, A.S. Mira, Finite Volume Methods for Pollutant Transport in Shallow Water Equation,, AIP Conference Proceedings, Vol 2293, 030023, 2020.

©UBMA – 2025

[12] D. Vanzo, A. Siviglia, E. F. Toro, Pollutant transport by shallow water equations on unstruc- tured meshes: hyperbolization of the model and numerical solution via a novel flux split- ting scheme, journal=Journal of Computational Physics, Journal of Computational Physics, vol 321, 1–20, 2016, Elsevier.

[13] X. Wu, F. Yang, and D. Liang, Study of Pollutant Transport in Environmental Flows using Depth-Averaged Random Walk Method, vol 3, 2342-2350 (2018).

[14] E. Audusse, M.O. Bristeau, Transport of pollutant in shallow water a two time steps kinetic method, Vol. 37, No 2, 2003, pp. 389–416.

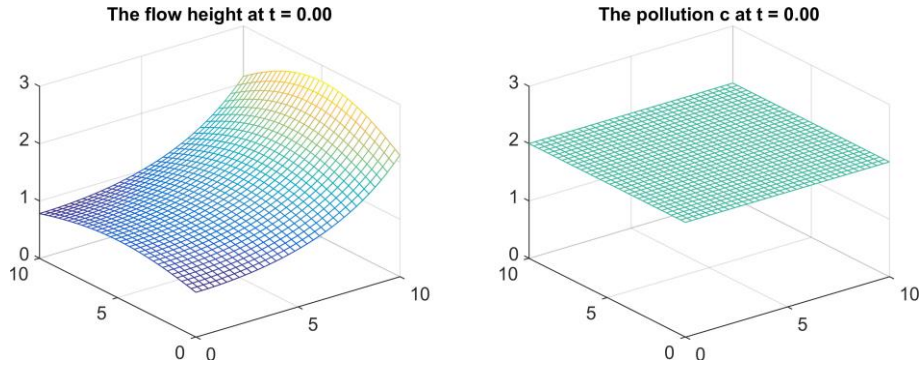


FIGURE 1 – Initially the flow section for the height *h* and pollution *c*.

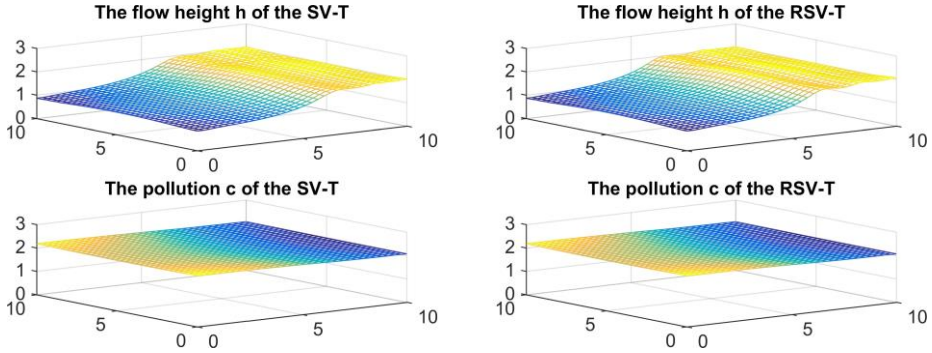


FIGURE 2 – Numerical solutions for the height *h* and pollution *c* for both the SV-T and the RSV- T, at time $t = 2$ with $\varepsilon = 10^{-3}$.

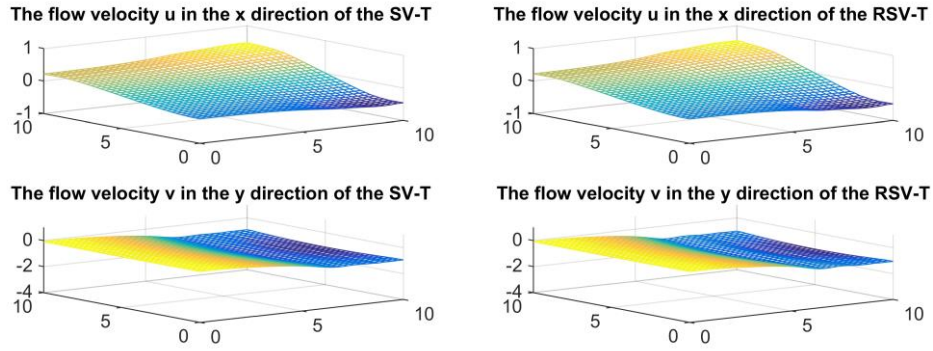


FIGURE 3 – Numerical solutions of the velocity (u, v) for the SV-T and the RSV-T, at time $t = 2$ with $\varepsilon = 10^{-3}$.

©UBMA – 2025

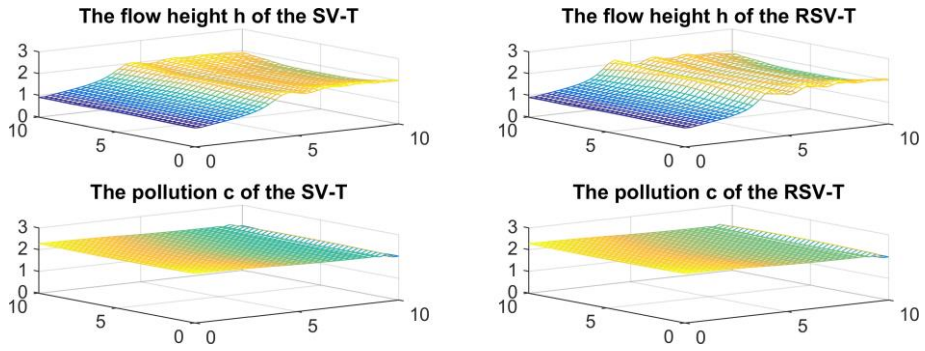


FIGURE 4 – Numerical solutions of the height h and pollution c for the SV-T and the RSV-T, at time $t = 3$ with $\varepsilon = 10^{-3}$.

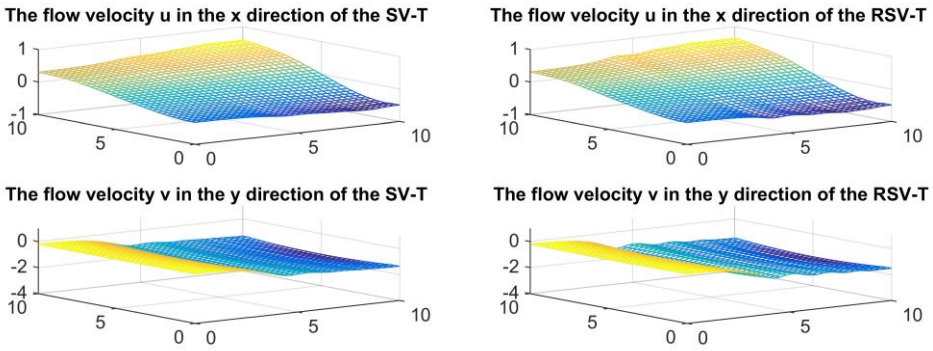


FIGURE 5 – Numerical solutions of the velocity (u, v) for the SV-T and the RSV-T, at time $t = 3$ with $\varepsilon = 10^{-3}$.

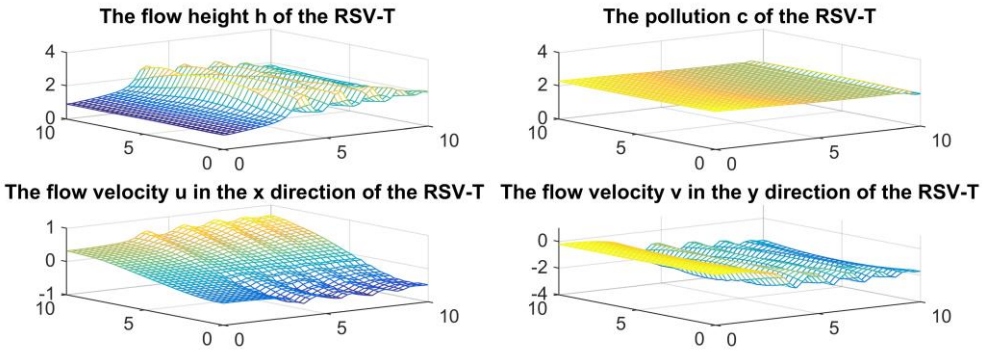


FIGURE 6 – Numerical solutions of the height h, pollution c and the velocity (u, v) for the RSV- T, at time $t = 3$ with $\varepsilon = 2 \cdot 10^{-3}$.

©UBMA – 2025