A multi-species population harvesting model with time and size competition dependence function

L. Louison, C. Nijman, A. Omrane

Université de Guyane, UMR EcoFoG (AgroParisTech, Cirad, CNRS, Inrae, Université des Antilles), Campus de Troubiran, 97337 Cayenne (FRANCE).

Article Info

Article history:

Received: 19/03/2025
Revised: 10/04/2025
Accepted: 20/04/2025

Keywords:
Ecosystems, Well-Posedness, Mathematical Modeling

Corresponding Author: *abdennebi.omrane@univ-guyane.fr*

ABSTRACT

We consider a multispecies nonlinear model describing a forest population which is height-structured. Here, each tree do compete for light with trees of bigger height. Using a fixed point argument, we prove the existence of a unique solution to the problem.

1. INTRODUCTION

Forests provide major ecosystem services to people around the world, and their conservation is very important. For this, we must avoid massive deforestation and also reduce forest degradation. Moreover, the maintenance of sustainable forest management is important for the carbon balance since it contributes to the mitigation of climate change.

We introduce a forest model to analyze the impact of different assumptions regarding vital rates (growth, mortality, recruitment), as well as different environmental conditions, on forest dynamics. Growth and mortality processes are influenced in a nonlinear way by competition for resources: light (mainly). A synthesis of works on the modeling of these problems can be found in several references. See for example: Calsina and Saldana [2], Chave [3] or Goetz *et* al. [4]. One can also see the references by Kato [6] and Kohyama [7].

The model for the population dynamics we propose is a generalization of the mathematical model for the homogeneous structured forest introduced by Ainseba *et* al. [1]. We study here the case of multi-species structured forest population by solving a system of Partial Differential Equations (PDE). The trees population is represented by a density function $u_s := u_s(t,h)$ population of species s, at time $t \in [0,T]$ and of height $h \in [h_0, h_{\max}]$ (we will set below $h_0 = 0$ and $h_{\max} = H$ for simplicity).

2. SECTION 1

For each species s, where $s = 1, 2\ldots, N_S$, we consider the Cauchy problem:

$$\begin{array}{lll} \partial_t u_s + \partial_h (V_s(E,h)u_s) = -m_s(E,h)u_s & for & (t,h) \in Q :=]0,T[\times]0,H[, \\ u_s(t,0)V_s(E(t,0),0) = B_s(t) & for & t \in]0,T[, \\ u_s(0,h) = u_{s,0}(h) & for & h \in]0,H[, \end{array} \qquad (1)$$

©UBMA – 2025

where $E(t,h) := \sum_{s=1} E_s(t,h)$ with

$$E_s(t,h) = \int_h^H \alpha(h')\, u_s(t,h')\, dh'$$

representing the competition (environmental) function. It is the intra species competition function corresponding to species *s*. In (1),

$$B_s(t) = \int_0^H \beta_s(E(t,h),h) u_s(t,h), dh$$

represents the birth function for the species *s*.

The function $V_s(E, h) := V_s(E(t, h), h)$ is the growth speed of the tree species *s;* it depends on the competition function with the other species. In the following, we suppose that the growth speed function V_s is non-negative, bounded and Lipschitz continuous with respect to E_s and h with a Lipschitz constant V_L. In a same manner, we define the function $m_s(E, h) := m_s(E(t, h), h)$ as the mortality rate from species *s*.

2.1. The method of characteristics to simplify the IBVP

A general used technique to analyze PDE's is the classical method of characteristics (see the references by Gurtin and MacCamy [5] and by Webb [8]). We will parameterize the height as a function depending on time *t*. We have:

$$\frac{du_s}{dt}(t,h(t)) = -G_s(E(t,h(t)),h(t))u_s(t,h(t)), \qquad s = 1,2,\cdots,N_s. \tag{2}$$

Lemma 1 *Let $u_s(t, h)$ be a solution of the problem (1) along the characteristic curves such that $h = \varphi_{\tau,\eta}(t)$ and $s = 1, 2, \cdots, N_s$. Then for any $\eta \geq 0$ and $\tau > 0$ and $w \in [0, t]$, the function u_s has the following expression along the characteristic curves:*

$$u_s(t,h) = \begin{cases} \dfrac{B_s(\tau)}{V_s(E(\tau,0),0)} \exp\Big(-\displaystyle\int_\tau^t G_s(E(w,h),h)dw\Big), & for\ a.e.\ h \quad with \quad h \in [0, <\phi[, \\ u_{s,0}(\eta)\exp\Big(-\displaystyle\int_0^t G_s(E(w,h),h)dw\Big), & for\ a.e.\ h \quad with, \quad h \in [\phi, H]. \end{cases}$$

Proof. Let $\tilde{\varphi}(t)$ be the characteristic curve through the initial pair $(\tilde{\tau},\tilde{\eta})$. Consider the function $u_S(t,h)$ along this curve :$u_{s,\tilde{\varphi}}(t) := u_s\big(t, \varphi_{\tilde{\tau},\tilde{\eta}}\big)$.

Considering the problem along the characteristic curves, the relation (2) gives :

$$\begin{cases} \dfrac{du_{s,\tilde{\varphi}}}{dt}(t) = -G_s(E(t,\tilde{\varphi}(t)),\tilde{\varphi}(t))u_{s,\tilde{\varphi}}(t) & \text{for t} > \tilde{\tau}, \\ u_{s,\tilde{\varphi}}(\tilde{\tau}) = u_s(\tilde{\tau},\tilde{\eta}) & \text{for } t = \tilde{\tau}. \end{cases}$$

With the classical approach of nonlinear problems we obtain a general solution to problem (1) almost everywhere each u_s is considered along $\tilde{\varphi}(t)$ characteristic curve starting at $(\tilde{\tau},\tilde{\eta})$:

©UBMA – 2025

$$\tilde{u}_{s,\tilde{\varphi}}(t) = u_s(\tilde{\tau},\tilde{\eta})\exp\left(\int_{\tilde{\tau}}^{t} -G_s(E(w,\varphi_{\tilde{\tau},\tilde{\eta}}(w)),\varphi_{\tilde{\tau},\tilde{\eta}}(w))dw\right). \quad (3)$$

At this point we consider the two types of characteristic curves as introduced in statement. For the characteristic curve with initial point $(\tau,0)$ we apply the boundary condition in (1) and obtain :

$$u_s(t,\varphi_{\tau,0}(w)) = \frac{B_s(\tau)}{V_s(E(\tau,0),0)}\exp\left(-\int_{\tau}^{t} G_s(E(w,\varphi_{\tau,0}(w)),\varphi_{\tau,0}(w))dw\right).$$

Considering our solution, $u_s(t,h)$ along characteristic curves starting at $\tau = t_0$, we finally obtain:

$$u_s(t,h) = u_{s,0}(\eta)\exp\left(-\int_{0}^{t} G_s(E(w,\varphi_{0,\eta}(w)),\varphi_{0,\eta}(w))dw\right).$$

3. THE EXISTENCE OF A SOLUTION TO THE NONLINEAR MODEL

In this section we will proof the existence of a solution to our problem. From Lemma 1, we deduce that the existence of a solution u_s, $s = 1, 2, \ldots, N_s$, is equivalent to the existence of E and B_s. We have the:

Proposition 2 *Denote by* $P_0 = \int_0^H u_0(h)d(h)$ *the total population corresponding to the initial condition* u_0 *and by* $E_0 = \sum_{s=1}^{N_s} E_s^0(t,h)$ *with* $E_s^0(t,h) = \int_h^H \alpha(h')u_0(h')dh'$.

Let M be the Banach space:

$$M = \{f \in \mathscr{C}(Q), f(0,0) = E_0,\ ||f|| \le K\}\ ; \quad and\ K = max(E_0,P_0).$$

Then, the linear and bounded operator $\mathscr{F}: M \to \mathscr{C}(Q)$ *is a contraction. This means that problem* (1) *has a unique solution.*

Proof. We first prove that $\mathscr{F}$ maps M into itself. After simple computations we obtain:

$$\|\mathscr{F}(E(t,h)\| \le \left\|\beta^* P_0 NYT_0 e^{\beta^* T_0} + E_0\right\| \le K. \quad (4)$$

Now, we show that F is a contration. Consider two different competition functions, one along the characteristic function with initials (τ_1, η_1) and one along the characteristic function with initials (τ_2, η_2).
We define:

$E^{(1)} := E(t,\varphi_{\tau_1,\eta_1}(t))$ and $E^{(2)} := E(t,\varphi_{\tau_2,\eta_2}(t))$, and for simplicity for I = 1,2. We denote by:

$$\begin{aligned}
\alpha^{(i,1)} &:= \alpha(\varphi_{\tau_i,0}(t)),\\
\alpha^{(i,2)} &:= \alpha(\varphi_{0,\eta_i}(t)),\\
B_s^{(i)}(\tau) &:= B_s(\tau_i),\\
u_s^{(i)}(\eta) &:= u_{s,0}(\eta_i),\\
g_s^{(i,1)} &:= g_s(E_s(t,\varphi_{\tau_i,0}(t)),\varphi_{\tau_i,0}(t)),\\
g_s^{(i,2)} &:= g_s(E_s(t,\varphi_{0,\eta_i}(t)),\varphi_{0,\eta_i}(t)).
\end{aligned} \quad (5)$$

©UBMA – 2025

For the i^{th} competition function we write:

$$E^{(i)} = \sum_{s=1}^{N_s} \left(\int_0^t \alpha^{(i,1)} B_s^{(i)} \exp\left(- \int_\tau^t g_s^{(i,1)} dw \right) d\tau + \int_0^H \alpha^{(i,2)} u_s^{(i)} \exp\left(- \int_0^t g_s^{(i,2)} dw \right) d\eta \right).$$

Then we have:

$$\begin{aligned}
& \left| \sum_{s=1}^{N_s} E(t, \varphi_{\tau_1,\eta_1}(t)) - \sum_{s=1}^{N_s} E(t, \varphi_{\tau_2,\eta_2}(t)) \right| \\
\leq & \left| \sum_{s=1}^{N_s} \left(\int_0^t \alpha^{(1,1)}(t) B_s^{(1)}(\tau) \exp\left(\int_\tau^t g_s^{(1,1)}(E_s^{(1)}, w) dw \right) d\tau \right. \right. \\
& \left. + \int_0^H \alpha^{(1,2)}(t) u_s^{(1)}(\eta) \exp\left(- \int_\tau^t g_s^{(1,2)}(E_s^{(1)}, w) dw \right) d\eta \right) \\
& - \sum_{s=1}^{N_s} \left(\int_0^t \alpha^{(2,1)}(t) B_s^{(2)}(\tau) \exp\left(- \int_\tau^t g_s^{(2,1)}(E_s^{(2)}, w) dw \right) d\tau \right. \\
& \left. \left. + \int_0^H \alpha^{(2,2)}(t) u_s^{(2)}(\eta) \exp\left(- \int_0^t g_s^{(2,2)}(E_s^{(2)}, w) dw \right) d\eta \right) \right|.
\end{aligned}$$

Now we use the linearity of integrals to have the upper bound:

$$\begin{aligned}
& \sum_{s=1}^{N_s} \left| \int_0^t \left(\alpha^{(1,1)}(t) B_s^{(1)}(\tau) \exp\left(\int_\tau^t g_s^{(1,1)}(E_s^{(1)}, w) dw \right) \right. \right. \\
& \left. - \alpha^{(2,1)}(t) B_s^{(2)}(\tau) \exp\left(- \int_\tau^t g_s^{(2,1)}(E_s^{(2)}, w) dw \right) \right) d\tau \\
& + \int_0^H \left(\alpha^{(1,2)}(t) u_s^{(1)}(\eta) \left(- \int_0^t g_s^{(1,2)}(E_s^{(1)}, w) dw \right) \right. \\
& \left. \left. - \alpha^{(2,2)}(t) u_s^{(2)}(\eta) \exp\left(- \int_0^t g_s^{(2,2)}(E_s^{(2)}, w) dw \right) \right) d\eta \right|.
\end{aligned}$$

In the rest of the proof we focus on the inner part of the sum of terms. On both terms the property $a_1 b_1 - a_2 b_2 = a_2(b_1 - b_2) + b_1(a_1 - a_2)$ is used. The above expression is then bounded by:

$$\begin{aligned}
& \left| \left(\int_0^t \left[\alpha^{(2,1)}(t) B_s^{(2)}(\tau) \left(\exp\left(- \int_\tau^t g_s^{(1,1)}(E_s^{(1)}, w) dw \right) - \exp\left(- \int_\tau^t g_s^{(2,1)}(E_s^{(2)}, w) dw \right) \right) \right. \right. \\
& + \exp\left(- \int_\tau^t g_s^{(1,1)}(E_s^{(1)}, w) dw \right) \left(\alpha^{(1,1)}(t) B_s^{(1)}(\tau) - \alpha^{(2,1)}(t) B_s^{(2)}(\tau) \right) \left. \right] d\tau \\
& + \int_0^H \left[\alpha^{(2,2)}(t) u_s^{(2)}(\eta) \left(\exp\left(- \int_0^t g_s^{(1,2)}(E_s^{(1)}, w) dw \right) \exp\left(- \int_0^t g_s^{(2,2)}(E_s^{(2)}, w) dw \right) \right) \right. \\
& \left. \left. + \exp\left(- \int_0^t g_s^{(1,2)}(E_s^{(1)}, w) dw \right) \left(\alpha^{(1,2)}(t) u_s^{(1)}(\eta) - \alpha^{(2,2)}(t) u_s^{(2)}(\eta) \right) \right] d\eta \right) \right|.
\end{aligned}$$

We now apply the triangle inequality, we have: $|e^{-x} - e^{-y}| \leq |x - y|, \forall x, y \geq 0$, and use the Lipschitzian property from $g_s^{(1,1)}$ and $g_s^{(1,2)}$ upon the respective integral functions such that

©UBMA – 2025

$\left|\mathcal{F}(E^{(1)}\text{-}\mathcal{F}(E^{(2)})\right|$ is upper bounded by:

$$\begin{aligned}
&\int_0^t \alpha^{(2,1)}(t)B_s^{(2)}(\tau)\left|\int_\tau^t g_s^{(1,1)}(E_s^{(1)},w)dw-\int_\tau^t g_s^{(2,1)}(E_s^{(2)},w)dw\right|d\tau\\
&+\left|\int_0^t e^0\left(\alpha^{(1,1)}(t)B_s^{(1)}(\tau)-\alpha^{(2,1)}(t)B_s^{(2)}(\tau)\right)d\tau\right|\\
&+\int_0^H \alpha^{(2,2)}(t)u_s^{(2)}(\eta)\left|\int_0^t g_s^{(1,2)}(E_s^{(1)},w)dw-\int_0^t g_s^{(2,2)}(E_s^{(2)},w)dw\right|d\eta\\
&+\left|\int_0^H\left(\alpha^{(1,2)}(t)u_s^{(1)}(\eta)-\alpha^{(2,2)}(t)u_s^{(2)}(\eta)\right)d\eta\right|.
\end{aligned} \tag{6}$$

We finally obtain the contraction : $\left|\mathcal{F}(E^{(1)}(t,h))-\mathcal{F}(E^{(2)}(t,h))\right| \le \sum_{s=1}^{N_s} R_s\left|E_s^{(1)}-E_s^{(2)}\right| \le NR\left|E_s^{(1)}-E_s^{(2)}\right|$,

where $R_S = max\ (R_1,R_2,R_3,R_4)$ and $R = max(R_S)$ are positive constants.

4. CONCLUSION

We presented a nonlinear PDE model for trees population dynamics for several species of trees. The population is subject to vital variables (mortality, recruitment.) and to competition. Using the characteristic method and the fixed point theory, we show that the problem is well posed in the continuous framework.

REFERENCES

[1] Ainseba B., Louison L., Omrane A., A population harvesting model with time and size competition dependence function. Journal of Optim. Theory and Applications, 195, pp 647- 665 (2022).

[2] Calsina A., Saldana J., A model of physiologically structured population dynamics with a nonlinear individual growth rate. J. Math. Biol., 33, pp 335-364 (1995).

[3] Chave J., Study of structural, successional and spatial patterns in tropical rain forests using TROLL, a spatially explicit forest model. Ecol. Modelling, 124(2-3), pp 233-254 (1999).

[4] Goetz R. U., Hritonenko N., Mur R., Xabadia A., Yatsenko Y., Forest management for timber and carbon sequestration in the presence of climate change: The case of Pinus Sylvestris. Ecol. Economics, 88, pp 86-96 (2013).

[5] Gurtin M. E., MacCamy R. C., Non-linear age-dependent population dynamics. Arch. Ration. Mech. Anal., 54, pp 281-300 (1974).

[6] Kato N., A general model of size-dependent population dynamics with nonlinear growth rate. J. Math. Anal. Appl., 297, pp 234-256 (2004).

[7] Kohyama T., Simulating stationary size distribution of trees in rain forests. Annals of Botany, 68, pp 173-180 (1991).

[8] Webb G. F., Theory of nonlinear age-dependent population dynamics, Marcel Dekker Inc., New-York and Basel (1985).

©UBMA – 2025