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Comparative Study on Dynamic Behaviour of Thick Beams on Foundation: Winkler, Pasternak and Kerr Foundation

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ABSTRACT

A comparative study amongst different types of elastic foundation of a bi-directionally graded beam mounted on elastic foundation is carried out. A single parameter Winkler, two parameters Pasternak and three parameters Kerr foundation are considered as elastic foundation. Bi-directional variation of the beam material is considered along the both thickness and length directions following power law distribution function. The kinematics of the beam is defined following Timoshenko beam theory. To obtain the geometric nonlinear free vibration response the problem is solved in two stages. At initial stage the static part of the problem is solved under loaded condition. Small amplitude dynamic analysis is followed next on the pre-deflected beam to get large amplitude free vibrational response. Displacement based Rayleigh-Ritz principle is utilized to formulate the problem and employing a numerical method the problem is solved. New sets of results are prepared for various combinations of system parameters to investigate the effect of various types of elastic foundations on dynamic behaviour.

1 Introduction

Beam resting on foundation plays pivotal role in modern engineering for modelling various complex problems such as road pavement, railway tracks and constantly supported pipelines etc. However, accurate modelling of this foundation, which requires proper beam-foundation interaction, is indeed a challenging task. For efficient and precise representation, different types of foundation models are available. Among them, the continuous medium model comes out as more precise but it involves higher mathematical complicity in formulation. Even after considering simplified assumptions, exact solutions are hardly attainable. In contrast, mechanical models are undoubtedly less accurate, but theoretically simpler and easier to apply. Winkler [1] presented the first-born, yet most renowned and the most commonly found mechanical model, wherein the foundation is described by an array of closely arranged, linearly elastic and mutually independent vertical springs that offer resistive forces to vary directly according to the deflection of the supported member. Such a foundation model involves a

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single parameter. However, independency of the springs results in an incapability to establish the interaction between adjacent springs. This is considered to be a rather rudimentary approximation with respect to the exact behaviour of the foundation. To overcome this drawback, in addition to the 1st parameter of Winkler foundation a 2nd parameter by means of shear layer is introduced. The corresponding models are known as two-parameter elastic foundation models [2-4] and the most popular one is known as Pasternak foundation model. Even with the introduction of an additional parameter, the theoretical formulation of such foundation remains simple [5]. Therefore, two-parameter models are more accurate compare to Winkler model and less complex than continuous medium model.

Further improvements are possible with the introduction of a third parameter and three-parameter models are the most generalized of all elastic foundation. Such models use an additional parameter with the two-parameter models to make it more realistic and effective. Foundation models presented by Hetenyi, Reissner and Kerr [6] falls in this category. The Kerr model is of particular interest in this category. In Kerr foundation model, in-between two elastic Winkler layers, a shear layer is considered, which provides added flexibility and convenience to maintain the continuity of the transverse displacement at the boundaries between different loading points of the foundation [6]. The mathematical modelling of such foundation is not as complex as the continuous medium model. The current paper intended to present a comparative study highlighting the effect of different types of foundation on the dynamic system behaviour of functionally graded (FG) thick beams-foundation system. At this point it would be prudent to discuss about the status of presently available literature. Presence of huge amount of research articles in the field of FG beams on foundation signifies the importance of this domain. It is noticed that majority the research papers concentrate on single directional FG beam on elastic foundation [7-27]. Amongst these article majority consider the simplified Winkler type of elastic foundation [7-13]. Although in some instances the two parameter Pasternak foundation is considered [14-21], but research articles associated with the problem of three parameter Kerr foundation are limited in number [22-27]. However, there appears to be comparatively lesser focus on the topic of bi-directionally functionally graded (BFG) beams supported on foundation. Such a material is a special class of graded materials, where, gradation is provided along two mutually orthogonal axes [28]. The ability of withstanding high temperature gradients in multiple directions makes this material useful for components in aircraft, space shuttles, automotive system, nuclear reactors and other high-performance applications [29]. Owing to its probable futuristic applications, this domain has recently generated some interest and some researchers have taken up studies relating to foundation supported BFG beams. A few such studies are highlighted in the following paragraph.

Rajasekaran and Khaniki [30] took up the problem of tapered BFG micro-beam supported on a foundation of variable stiffness to investigate induced vibrations from a travelling mass. Linear, parabolic, exponential, and sigmoidal distribution functions were used for the bi-directional material gradation. The principle of couple stress theory was utilized to represent the size dependent characteristic of the micro-beam. A similar kind of problem was taken up by Azandariani et al [31] on Euler–Bernoulli BFG micro-beam which was assumed on a nonlinear elastic support to study nonlinear static behaviour. The differential equations were deduced using the same method as discussed by Rajasekaran and Khaniki [30]. However, for solution of the problem the authors utilized the numerical scheme of generalized differential quadrature method. Recently, Vu et al [32] performed dynamic analysis on a partially Pasternak foundation supported BFG sandwich beams loaded under a travelling mass. The material of the face layers was considered as BFG material, while the core was considered an axially graded material. A quasi-3D theory which apply the principle of finite element analysis was employed for deriving the underlying equations.

It is noticeable from the literature survey that the papers usually focus on individual foundation type for the respective analysis. A comparative study in regards to the effect of various types of foundation on the system is as yet unreported. In the present article, the large amplitude free vibrational responses of bi-directionally graded thick beam resting on three dissimilar kinds of elastic foundation are investigated and compared. As already mentioned, Winkler, Pasternak and Kerr type of foundation models are utilised in the present work. The system parameter whose variation is observed with the change in foundation is the natural frequencies of the externally loaded beam, which in this situation is a bi-directionally graded thick beam. A displacement-based technique which utilizes the Ritz approximation to assume the displacement fields is implemented for formulating the current problem. The derivation of the governing differential equations is conducted by following suitable energy principle, while the numerical scheme of direct substitution method is applied for solving the derived governing differential equations. It is to be noted that the present analysis is a whole domain analysis in which suitable reference points are created to perform the detailed computations.

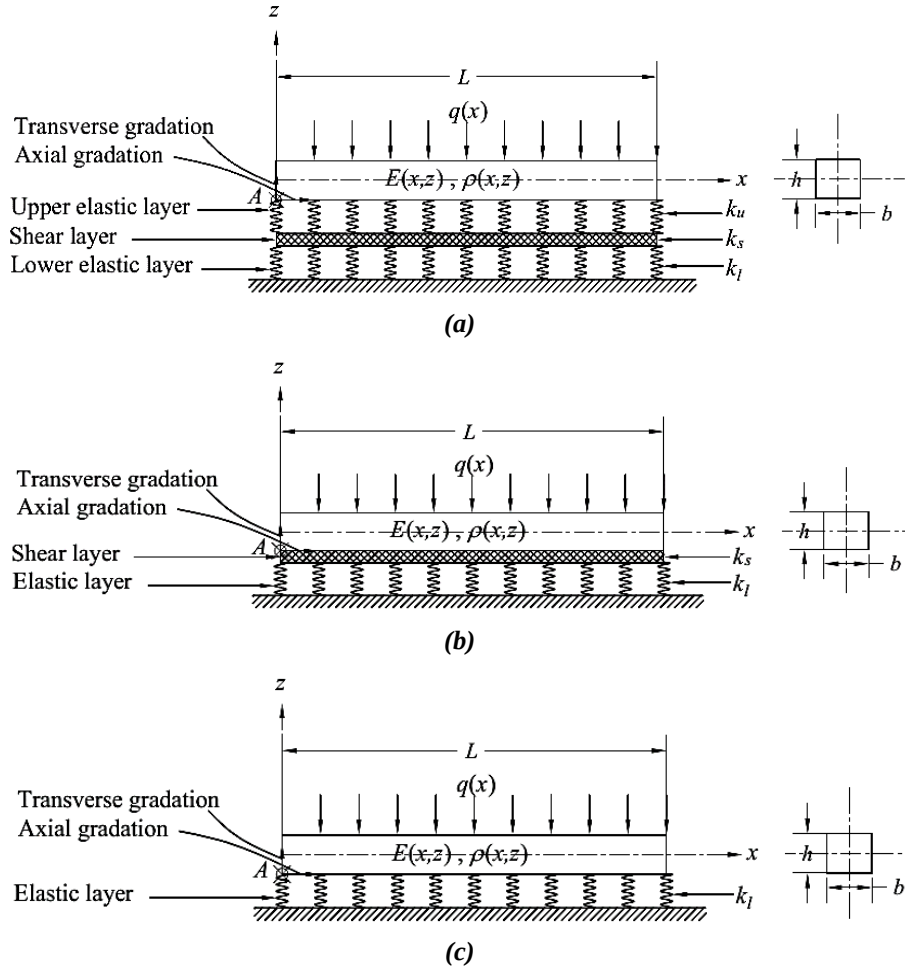


Fig. 1 – Representation of Bi-directional graded (BFG) beam on (a) Kerr (b) Pasternak and (c) Winkler foundation, respectively.

2 Mathematical formulation

Fig. 1 illustrates a BFG prismatic beam of length L and having rectangular cross-sectional shape (width b and thickness h), supported on an elastic foundation and loaded with an evenly distributed load, $q(x)$. To represent the elastic foundation three different models are used and these are Kerr, Pasternak and Winkler foundation, respectively. Fig. 1(a) represents a Kerr foundation, which is the most appropriate and generalized form of elastic foundation model. It is a three-parameter foundation where three prominent layers are present. The lower elastic layer rests on the rigid ground, which is followed by an intermediate shear layer and the upper layer again is an elastic layer that connects the shear layer to the beam. Both the elastic layers are modelled using a series of mutually independent linear springs with fixed stiffness values of K_l and K_u for lower and upper elastic layers, respectively. Whereas, the shear layer is used to counter the coupling of springs which is specified by a fixed torsional stiffness value of K_s . By eliminating the upper elastic layer, the Kerr foundation is reduced to Pasternak foundation, which is shown in Fig. 1(b). Likewise, when both the intermediate shear layer and the upper layer are discarded, the Kerr foundation is further reduced to Winkler foundation (Fig. 1(c)). For To perform bi-directional material gradation in beam the edge containing point 'A' is selected as the reference and from this edge a smooth functional transition of material is assumed along x and z -coordinate direction in accordance with power law distribution function [33], mathematically expressed in Eq. (1). The material property variation of elasticity (E), density (ρ) and Poisson's ratio (μ) is controlled in such a way that at root edge the material will be metal (Aluminium) rich, while at the opposite edge the material is converted to ceramic (Alumina) rich [34, 35],

$$P(x, z) = P_m + (P_c - P_m) \left(\frac{x}{L} \right)^{n_x} \left(\frac{z}{h} \right)^{n_z} \quad (1)$$

In Eq. (1), P represents material properties (E , ρ and μ) as a single abbreviation. P_m is material property value at the root edge, where the material behaves as pure metal. Whereas, P_c denotes material property value at the opposite edge, where the material behaves as pure ceramic. Here, n_x and n_z are gradation indexes which control the material distribution along the axial and transverse direction, respectively. When these parameters are zero ($n_x = 0$ and $n_z = 0$), it represents the case of homogeneous material with the constant material properties of pure metal. Non-zero values represent the case of BFG material. Furthermore, the cases where $n_x = 0$ and $n_z \neq 0$, or $n_x \neq 0$ and $n_z = 0$, represent transversely and axially functionally graded (TFG and AFG) materials, respectively.

The response of the system is compared in terms of the free vibration response under large static displacement of the beam. So, it is essential to determine the free vibration frequencies of the system under static loading leading to large deformation. To achieve the desired objective, two separate but connected steps are followed to formulate the problem. The 1st step is solely dedicated to static analysis, which is performed under large amplitude stationary loading to produce large deformation. It is to be noted that this large amplitude nature of deformation makes the strain-displacement relation nonlinear and results in geometric nonlinearity. In the next step, the static solution of the beam is utilized to conduct the free vibration analysis (dynamic analysis) for determining the modal characteristics of the loaded beam. To formulate, both static and dynamic segments, a displacement based meshless technique is adopted, where, appropriate energy principles are used to deduce the governing equations. The static solution is attained by means of numerical implementation of an iterative scheme, while the dynamic problem is solved as an eigen value problem.

The kinematics of the beam is illustrated following Timoshenko beam theory (TBT). Therefore, the in-plane and transverse displacements ($\bar{u}$ and $\bar{w}$) at an arbitrary point of z distance from neutral surface can be given as,

$$\bar{u}(x, z) = u(x) - z\psi(x) \quad (2a)$$

$$\bar{w}(x, z) = w(x) \quad (2b)$$

In Eq. (2), u and w are in-plane (axial) and transverse displacements fields, respectively, defined at neutral surface, while, ψ is rotational displacement field about neutral axis. They (u , w and ψ) are functions of axial coordinate only.

Incorporating the von Kármán type of geometric nonlinearity, the expression of the axial strain (ϵ_{xx}) and shear strain (γ_{xz}), is presented in Eq. (3). Once Eq. (3) is substituted in linear elastic relations, the expression of normal and shear stresses (σ_{xx} and τ_{xz}) are obtained, as expressed in Eq. (4).

$$\epsilon_{xx} = (u') + \frac{1}{2}(w')^2 - z(\psi') \quad (3a)$$

$$\gamma_{xz} = \frac{1}{2}[(w') - \psi] \quad (3b)$$

$$\sigma_{xx} = E(x, z) \left[(u') + \frac{1}{2}(w')^2 - z(\psi') \right] \quad (4a)$$

$$\tau_{xz} = 2G(x, z)\gamma_{xy} = G(x, z)[(w') - \psi] \quad (4b)$$

It is to be highlighted that in the entire formulation the 1st derivatives w.r.t. space (x) and 1st derivatives w.r.t. time (τ) are represented by 'prime' and 'over-dot' symbol, respectively.

Substituting the above expressions of strains and stresses (Eqs. (3) and (4)) in strain energy relations, the axial and shear strain energy (U_{ax} and U_{sh}) can be obtained as,

$$U_{ax} = \frac{1}{2} \int_0^L \left[\left\{ \frac{1}{4}(w')^4 + (u')^2 + (w')^2 (u') \right\} E_1(x) - \left\{ (w')^2 (\psi') + 2(u')(\psi') \right\} E_2(x) + (\psi')^2 E_3(x) \right] dx \quad (5)$$

$$U_{sh} = \frac{k_{sh}}{2} \int_0^L \left[(w')^2 - 2(w')\psi + \psi^2 \right] G_1(x) dx \quad (6)$$

In Eq. (6), k_{sh} is shear correction factor, is introduced to take care of uneven distribution of shear strain across cross-section. The beam under consideration is bi-directionally graded. Therefore, the stiffness coefficients used in the above equations (Eqs. (5) and (6)) are function of axial coordinate, which can be stated as,

$$\begin{aligned} [E_1(x), E_2(x), E_3(x)] &= b \int_{-\frac{h}{2}}^{+\frac{h}{2}} [1, z, z^2] E(x, z) dz \\ G_1(x) &= b \int_{-\frac{h}{2}}^{+\frac{h}{2}} G(x, z) dz \end{aligned}$$

The energy stored in the foundation has significant contribution to the overall energy of the system. The strain energy of Kerr foundation under transverse deflection of beam, can be obtained as [36-39],

$$U_K = \frac{1}{2} \int_0^L \left[\left(\frac{k_l k_u}{k_l + k_u} \right) w^2 + \left(\frac{k_s k_u}{k_l + k_u} \right) (w')^2 \right] dx \quad (7)$$

By eliminating the upper elastic layer (of stiffness K_u), the Kerr foundation is reduced to Pasternak foundation (refer Fig. 1(b)), whose strain energy expression can be given as,

$$U_P = \frac{1}{2} \int_0^L \left[(k_l) w^2 + (k_s) (w')^2 \right] dx \quad (8)$$

Likewise, when both the intermediate shear layer (of stiffness K_s) and the upper layer (of stiffness K_u) are eliminated, the Kerr foundation is further reduced to Winkler foundation (refer Fig. 1(c)). The strain energy of such foundation is rewritten as,

$$U_W = \frac{1}{2} \int_0^L k_l w^2 dx \quad (9)$$

Finally, overall strain energy of the system (U) can be calculated by summing up both the axial and shear strain energy of beam with the strain energy stored in the foundation, as expressed in Eq. (10). Depending on the nature of foundation the expression of U_F can be changed to either U_K or U_P or U_W .

$$U = U_{ax} + U_{sh} + U_F \quad (10)$$

Furthermore, the potential energy (V) and the kinetic energy (T) of the loaded dynamic system can be evaluated as,

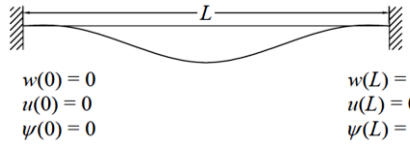
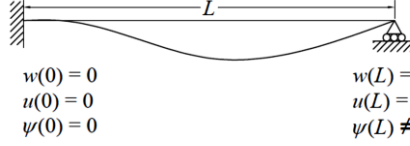
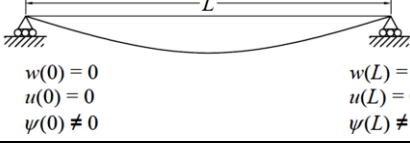
$$V = \int_0^L q(x) w dx \quad (11)$$

$$T = \frac{1}{2} \int_0^L \left[\left\{ (\dot{w})^2 + (\dot{u})^2 \right\} \rho_1(x) + (\dot{\psi})^2 \rho_3(x) \right] dx \quad (12)$$

Similar to the stiffness coefficients, the inertia coefficients of Eq. (12) are also function of axial coordinate, given as,

$$[\rho_1(x), \rho_3(x)] = b \int_{-\frac{h}{2}}^{+\frac{h}{2}} [1, z^2] \rho(x, z) dz$$

Table 1 - Details of start functions for various configurations of beam.

BCs	Beam configuration	Displacement field	Function
CC		w	$\phi_1 = (x/L)\{1-(x/L)\}$
		u	$\alpha_1 = (x/L)\{1-(x/L)\}$
		ψ	$\beta_1 = \sin(\pi x/L)$
CS		w	$\phi_1 = (x/L)\{1-(x/L)\}$
		u	$\alpha_1 = (x/L)\{1-(x/L)\}$
		ψ	$\beta_1 = \sin(\pi x/2L)$
SS		w	$\phi_1 = (x/L)\{1-(x/L)\}$
		u	$\alpha_1 = (x/L)\{1-(x/L)\}$
		ψ	$\beta_1 = \cos(\pi x/L)$

2.1 Static analysis

The static analysis is carried out with large displacement assumption for externally applied loading to determine the static deformation of beam. Being a displacement-based formulation, the static displacement fields (w , u and ψ) are approximated in terms of orthogonal functions (ϕ_i , α_i and β_i for w , u and ψ , respectively) and unknown parameters (d_i) using Rayleigh-Ritz method, as,

$$w(x) = \sum_{i=1}^{nw} d_i \phi_i(x) \quad (13a)$$

$$u(x) = \sum_{i=1}^{nu} d_{nw+i} \alpha_i(x) \quad (13b)$$

$$\psi(x) = \sum_{i=1}^{nsi} d_{nw+nu+i} \beta_i(x) \quad (13c)$$

Here, ϕ_i , α_i and β_i are selected fulfilling certain conditions: (i) They must be continuous and differentiable in the analysis domain, (ii) They must satisfy at least the essential boundary conditions (BCs) of the beam. The 1st set (ϕ_1 , α_1 and β_1) of these functions is known as start functions, are tabulated in Table 1 for different BCs [40]. From these start functions other higher order functions (upto the number of nw , nu and nsi) extracted following the scheme of Gram-Schmidt orthogonalization.

Static analysis utilizes the potential energy principle to deduce governing set of equations, which is expressed mathematically as,

$$\delta(U+V) = 0 \quad (14)$$

The appropriate substitution of energy expressions (Eqs. (10) and (11)) and assumed displacement fields (Eq. (13)) into Eq. (14), the governing equations can be rewritten in matrix form in terms of stiffness matrix $[K]$, vector of unknown parameters $\{d\}$ and load vector $\{f\}$ as,

$$[K]\{d\} = \{f\} \quad (15)$$

In Eq. (15), $[K]$ is a symmetric matrix of order $(nw + nu + nsi)$. The vector $\{d\}$ and $\{f\}$ appear as column matrix with dimensions of $(nw + nu + nsi) \times 1$. The specific structures and components of $[K]$ and $\{f\}$ can be found in the Appendix. A close look on $[K]$ reveals that few elements of it are functions of d , which makes the evaluation of $\{d\}$ complicated. Hence, an indirect iterative method of direct substitution is utilized for solving the problem [41]. Once $\{d\}$ is known, the static deflection configuration of the beam can be evaluated by substituting in Eq. (13).

2.2 Dynamic analysis

The dynamic analysis is studied on the statically solved beam configuration to examine the large amplitude modal characteristics. The dynamic displacement fields (w , u and ψ ,) are approximated in terms of orthogonal functions (ϕ_i , α_i and β_i for w , u and ψ , respectively) and unknown parameters (c_i) following Rayleigh-Ritz approximation as,

$$w(x, \tau) = \sum_{i=1}^{nw} c_i \phi_i(x) e^{j\omega\tau} \quad (16a)$$

$$u(x, \tau) = \sum_{i=1}^{nu} c_{nw+i} \alpha_i(x) e^{j\omega\tau} \quad (16b)$$

$$\psi(x, \tau) = \sum_{i=1}^{nsl} c_{nw+nu+i} \beta_i(x) e^{j\omega\tau} \quad (16c)$$

Eq. (16) shows that dynamic displacement fields hold coordinate (x) and time dependent parts (τ). Both are assumed to be completely separable. Out of these, axial coordinate dependent part exactly coincides with static displacement fields with ϕ_i , α_i and β_i are identical as that of static problem. However, c_i s are a fresh set of unknown parameters. ω is system's circular frequency and j indicates to $\sqrt{-1}$.

To deduce the governing set of equations Hamilton's principle is employed, which shows,

$$\delta \left(\int_{\tau_1}^{\tau_2} [T - (U + V)] d\tau \right) = 0 \quad (17)$$

In Eq. (17), τ denotes time. Already stated that the present dynamic analysis is carried out on statically solved system, whose deformation profile is already determined in the previous step. So, in dynamic part, U remains same as that of Eq. (10) and V needs to be zero.

The appropriate substitution of energy functionals (Eqs. (10) and (12)) and displacements field of Eq. (16) into Eq. (17), final expression in matrix form comes out to be,

$$[K]\{c\} - \omega^2 [M]\{c\} = 0 \quad (18)$$

In Eq. (18), $[K]$ is stiffness matrix which eventually appears to be same as that of static analysis and $[M]$ is mass matrix. The details of $[M]$ are elaborated in Appendix.

3 Results and discussions

This study explores large amplitude free vibration response of BFG beam supported by various types of elastic foundation. Geometric nonlinearity emerges from large deformation is included through the representation of nonlinear strain-displacement relation. For effective modelling of the present system three dissimilar classes of foundation models are considered. These are Kerr, Pasternak and Winkler foundation model, respectively. To analyse the influence of these foundations on the system response, results generated from each system are compared with the results of others. The material gradation in BFG beam is achieved along the longitudinal and transverse thickness directions as per the power law distribution function as outlined in Eq. (1). Elasticity, density and Poisson ratio at origin are specified as $E_m = 70$ GPa; $\rho_m = 2700$ kg/m³; $\mu_m = 0.3$, respectively, which matches the property of Aluminium (Al) [34]. The corresponding value of these material properties at the opposite edge w.r.t. the origin edge is taken as $E_c = 380$ GPa; $\rho_c = 2960$ kg/m³; $\mu_c = 0.3$, respectively, which matches the property of Alumina (Al₂O₃) [34]. The beam dimensions are selected as, $L = 1$ m, $b = 0.05$ m, $h = 0.01$ m and the following non-dimensional terms are utilized for generating results,

$$\xi = \frac{x}{L}, \eta = \frac{z}{h}, a_r = \frac{L}{h} \text{ and } \lambda = \omega \left(\frac{L^2}{h} \right) \sqrt{\frac{\rho_m}{E_m}}$$

The present investigation is performed as whole domain analysis. For detailed computation, the entire domain is transformed into a normalized one and an appropriate quantity of guiding points are produced along the axial ($\xi = x/L$) and transverse ($\eta = z/h$) directions. These points are known as gauss points. The choice of quantity of gauss points (ngx/ngz) are to be done precisely and carefully. Along with these, the selection of quantity of the orthogonal functions ($nw/nu/ksi$) is also crucial and must be done with care. The computational efficacy and correctness heavily rely on the appropriate choice of these parameters. Hence, it is important to conduct a suitable convergence study for selecting these parameters. After conduction of a thorough convergence analysis, the quantities are selected as $ngx/ngz = 24$ and $nw/nu/ksi = 8$.

For validating the present methodology, natural frequencies of the initial four modes are contrasted with the results of Jena et al [36] for a thick TFG SS beam, varying gradation index values, as shown in Table 2. To carry out this comparison, the present system is simplified to match the system of the literature [36] by assuming all the parameter values same as of the literature. In addition, a separate comparison study is performed with the published results of Simsek [42], which considers a BFG Timoshenko beam. The non-dimensional 1st frequencies ($\lambda_1 = \omega_1 \left(\frac{L^2}{h} \right) \sqrt{\frac{\rho_m}{E_m}}$) are compared for different boundary conditions and material indexes (n_x & n_z), as shown Table 3(a)-3(b). In both the cases the comparison shows that generated results are close to each other with the recognized results.

Table 2 - Comparison of first four natural frequencies (f in kHz) with the results of Jena et al [36] for transversely graded Timoshenko SS beam.

n	Literatures	f_1	f_2	f_3	f_4
0	Jena et al [36]	06.2450	12.4901	18.7351	24.9802
	Present Study	06.0504	12.1031	18.1514	24.1833
1	Jena et al [36]	03.7618	07.5225	11.2808	15.0355
	Present Study	03.6030	07.2074	10.8089	14.4004
5	Jena et al [36]	02.9433	05.8863	08.8286	11.7698
	Present Study	02.7917	05.5846	08.3753	11.1582
10	Jena et al [36]	02.7930	05.5859	08.3786	11.1708
	Present Study	02.6424	05.3641	07.9274	10.5617

Table 3(a) - Comparison of first natural frequencies (λ_1) with results of Simsek [42] for BFG Timoshenko SS beam.

SS Beam	$a_r=5$		n_z				
	n_x	0.0	0.2	0.4	0.6	0.8	1.0
Simsek [42]	0.0	2.6767	2.6748	2.6669	2.6533	2.6337	2.6103
Present study		2.6727	2.6705	2.6693	2.6688	2.6687	2.6686
Simsek [42]	0.2	2.6767	2.6728	2.6650	2.6513	2.6318	2.6083
Present study		2.6695	2.6670	2.6654	2.6644	2.6638	2.6631
Simsek [42]	0.4	2.6728	2.6689	2.6611	2.6474	2.6279	2.6044
Present study		2.6653	2.6624	2.6603	2.6587	2.6574	2.6559
Simsek [42]	0.6	2.6650	2.6630	2.6552	2.6416	2.6220	2.5986
Present study		2.6599	2.6566	2.6540	2.6517	2.6496	2.6471
Simsek [42]	0.8	2.6572	2.6533	2.6455	2.6318	2.6123	2.5888
Present study		2.6535	2.6497	2.6464	2.6433	2.6402	2.6366
Simsek [42]	1.0	2.6455	2.6416	2.6337	2.6201	2.6005	2.5771
Present study		2.6458	2.6415	2.6374	2.6334	2.6292	2.6242

Table 3(b) - Comparison of first natural frequencies (λ_1) with results of Simsek [42] for BFG Timoshenko CS beam.

CS Beam	$a_r=5$		n_z				
	n_x	0.0	0.2	0.4	0.6	0.8	1.0
Simsek [42]	0.0	3.8779	3.8759	3.8662	3.8505	3.8310	3.8037
Present study		3.8674	3.8546	3.8352	3.8090	3.7761	3.7370
Simsek [42]	0.2	3.8232	3.8212	3.8115	3.7958	3.7763	3.7490
Present study		3.8064	3.7932	3.7732	3.7466	3.7133	3.6735
Simsek [42]	0.4	3.7685	3.7666	3.7568	3.7431	3.7216	3.6962
Present study		3.7450	3.7312	3.7108	3.6836	3.6499	3.6095
Simsek [42]	0.6	3.7138	3.7099	3.7021	3.6865	3.6669	3.6416
Present study		3.6829	3.6686	3.6476	3.6201	3.5858	3.5450
Simsek [42]	0.8	3.6572	3.6552	3.6455	3.6318	3.6123	3.5869
Present study		3.6201	3.6053	3.5838	3.5556	3.5208	3.4796
Simsek [42]	1.0	3.6005	3.5986	3.5908	3.5751	3.5556	3.5322
Present study		3.5566	3.5412	3.5191	3.4903	3.4550	3.4132

Non-dimensional fundamental natural frequency (λ_1) of Kerr foundation supported beam for various combination of material gradation indexes (n_x and n_z) are shown in Fig. 2.

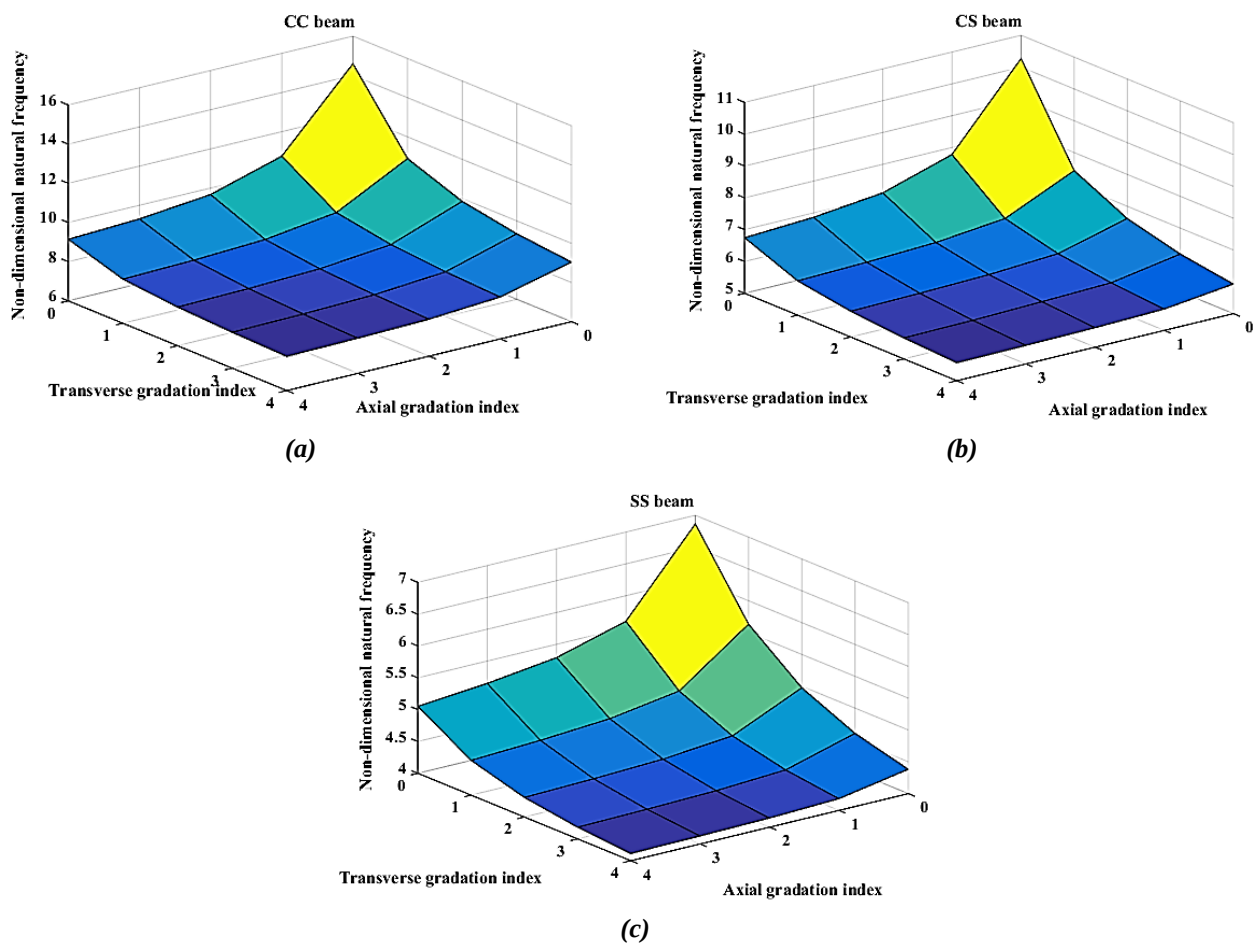


Fig. 2 - Non-dimensional fundamental natural frequency (λ_1) of Kerr foundation supported beam for different combination of material gradation indexes (n_x and n_z) for (a) CC (b) CS and (c) SS beam.

Three different plots namely, (a), (b) and (c), depicted from there dissimilar BCs are presented. In each plot, five different values of gradation indices are selected at a range from 0 to 4 with a fixed interval of 1. The stiffness value of each foundation layer is kept as constant. The stiffness values k_u & k_l are considered as 5 kPa, while stiffness k_s is taken as 5 kN. From the figures, it is detected that for a fixed value of gradation indices (n_x and n_z), natural frequency is maximum for CC beam and minimum for SS beam. It can also be noticed that for a particular BC natural frequency is maximum for homogeneous beam ($n_x = n_z = 0$) and minimum for maximum value of material gradation indices ($n_x = n_z = 4$). It is also prominent that for a given BC and fixed value of n_z natural frequency decreases as n_x increases. Similar pattern is observed after n_z is varied and n_x is reserved as constant.

Natural frequencies of the 1st and 2nd vibrational modes (λ_1, λ_2) for different cases of elastic foundations and foundation stiffness values are presented in Table 4. Here also results are presented for all three boundaries conditions. Under each BCs, three various types of elastic foundation (Kerr, Pasternak and Winkler foundation) and their four different values of foundation stiffness are considered. $k_u = k_l$ value is varied from 0 kPa to 30 kPa, while, for intermediate shear layer, k_s is varied from 0 kN to 30 kN. The foundation with '0' value of stiffness in all comprising layers indicates beam without any form of foundation. However, in such scenario, Eq. (7) provides an undefined value of strain energy. For reason being, a higher order term which is tends to zero (≈ 0), is used for stiffness of all comprising layers to define beam without foundation. The gradation indices ($n_x = n_z$) are kept as fixed at 1. According to Table 4, for every possible combination, an increase in foundation stiffness value leads to a higher natural frequency. The dependency of natural frequency on BC is quite prominent from the results. For every possible combination, CC beam exhibits highest natural frequency, while SS beam shows lowest. The results tabulated in Table 4 considers equal values for k_u (kPa), k_l (kPa) and k_s (kN) i.e., k_u (kPa) = k_l (kPa) = k_s (kN). This kind of consideration results the overall stiffness for the Pasternak foundation is more compare to Kerr foundation, resulting in more foundation stain energy in Pasternak foundation than the Kerr foundation, which is evident from Eq. (7) and Eq. (8). Because of this, the Pasternak foundation shows higher natural frequencies than the Kerr foundation across all stiffness levels, which is evident from the table. As there present no shear layer in Winkler foundation, the overall stiffness, as well as the stored strain energy is less in this foundation (Eq. (9)), which results in the lowest value in natural frequency.

Table 4 - Natural frequencies of first two vibrational modes (λ_1 and λ_2) for different types of elastic foundations and foundation stiffness values.

BCs	k_u (kPa) = k_l (kPa) = k_s (kN)	Kerr foundation		Pasternak foundation		Winkler foundation	
		k_u, k_l, k_s		k_l, k_s		k_l	
		λ_1	λ_2	λ_1	λ_2	λ_1	λ_2
CC	00	08.4885	23.4576	08.4885	23.4576	08.4885	23.4576
	10	09.5137	24.8167	10.4300	26.1000	08.6518	23.5172
	20	10.4300	26.1000	12.0366	28.4812	08.8121	23.5767
	30	11.2651	27.3186	13.4344	30.6642	08.9695	23.6359
CS	00	05.7793	18.9208	05.7793	18.9208	05.7793	18.9208
	10	07.0661	20.4325	08.1457	21.8385	06.0158	18.9945
	20	08.1457	21.8385	09.9478	24.4049	06.2434	19.0680
	30	09.0934	23.1579	11.4596	26.7214	06.4630	19.1411
SS	00	03.9884	28.3499	03.9884	28.3499	03.9884	28.3499
	10	05.5732	30.0219	06.7980	31.5850	04.3248	28.3997
	20	06.7980	31.5850	08.7471	34.4488	04.6368	28.4494
	30	07.8335	33.0562	10.3347	37.0384	04.9291	28.4990

The effect of different types of elastic foundation on large amplitude free vibration response is presented by plotting the 1st backbone curves (BBCs) in a normalized frequency-amplitude plane, as shown in Figs. 3-5. In every case, three separate subplots are presented, one for each of three BC. For all subplots, gradation indexes are taken as $n_x = n_z = 1$. Fig. 3 represents the results of Kerr foundation supported beam. Here, stiffness values k_u & k_l are varied from 0 kPa to 30 kPa, while, stiffness k_s is varied from 0 kN to 30 kN.

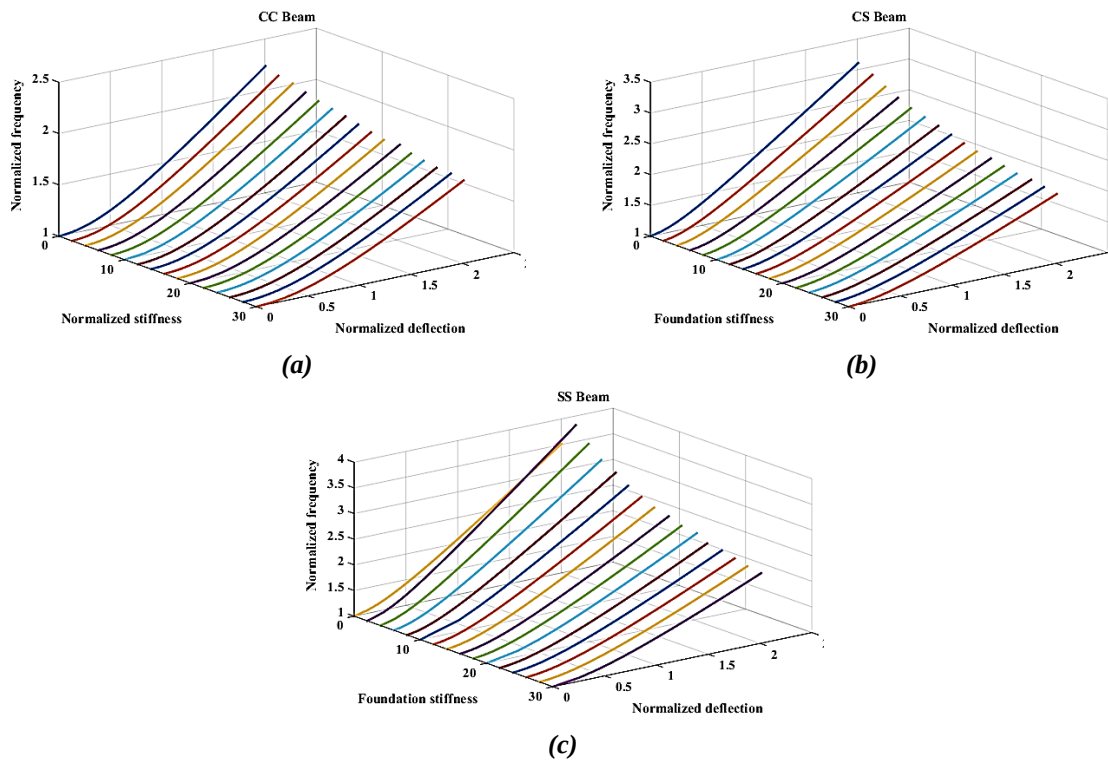


Fig. 3 - 1st BBCs for various combination of foundation stiffness values of Kerr foundation supported BFG (a) CC (b) CS and (c) SS beam.

Fig. 4 represents the results of Pasternak foundation. Here, the upper elastic layer is absent, so the stiffness values of remaining layers are k_s and k_t , are varied from 0 kPa to 30 kPa and 0 kN to 30 kN, respectively.

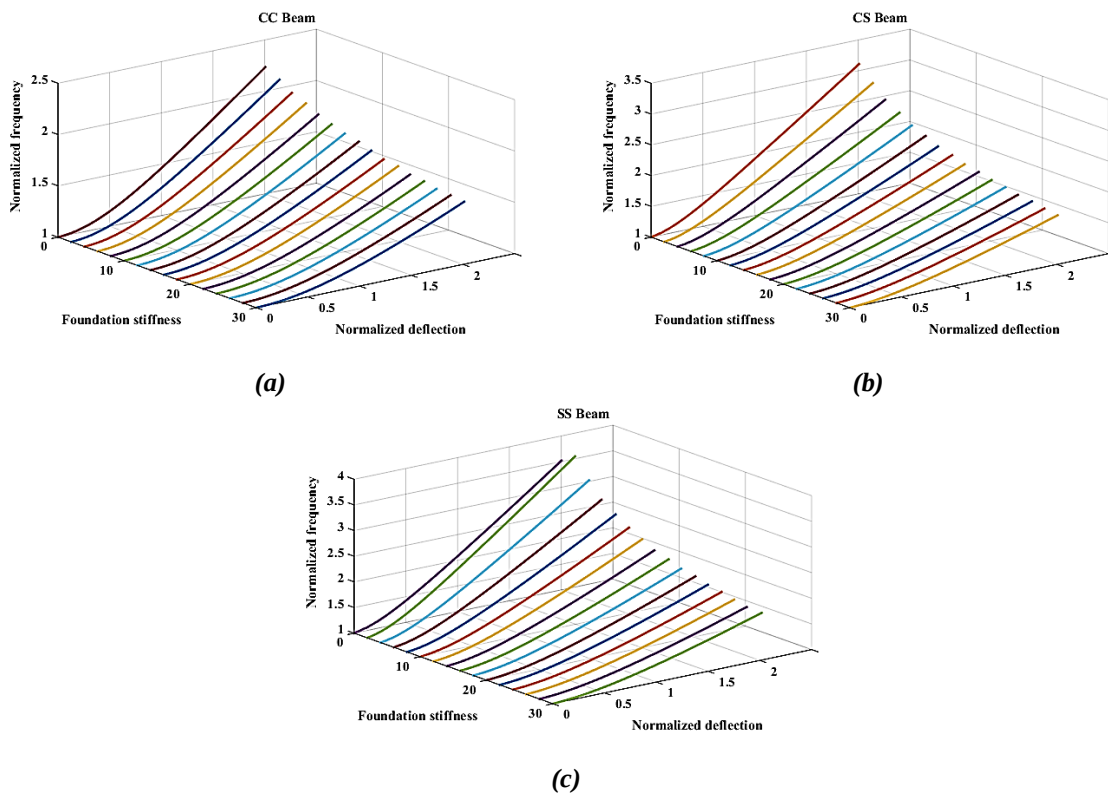


Fig. 4 - 1st BBCs for various combination of foundation stiffness values of Pasternak foundation supported BFG (a) CC (b) CS and (c) SS beam.

Fig. 5 represents the results of Winkler foundation, in which only the lower elastic layer is present. The stiffness of lower elastic layer k_l is varied from 0 kPa to 30 kPa. It is clearly understood that $k_u = k_s = k_l = 0$ represents the scenario of beam without any foundation support, is also included in the result to check the effectiveness of elastic foundation. From the results, it is quite evident that high value of stiffness leads to higher slope in BBC. This is a rather predictable outcome and can be described in a way that increasing foundation stiffness elevates the overall stiffness of the system which eventually leads to higher natural frequencies, and these high values of natural frequencies increase the slope of the BBC.

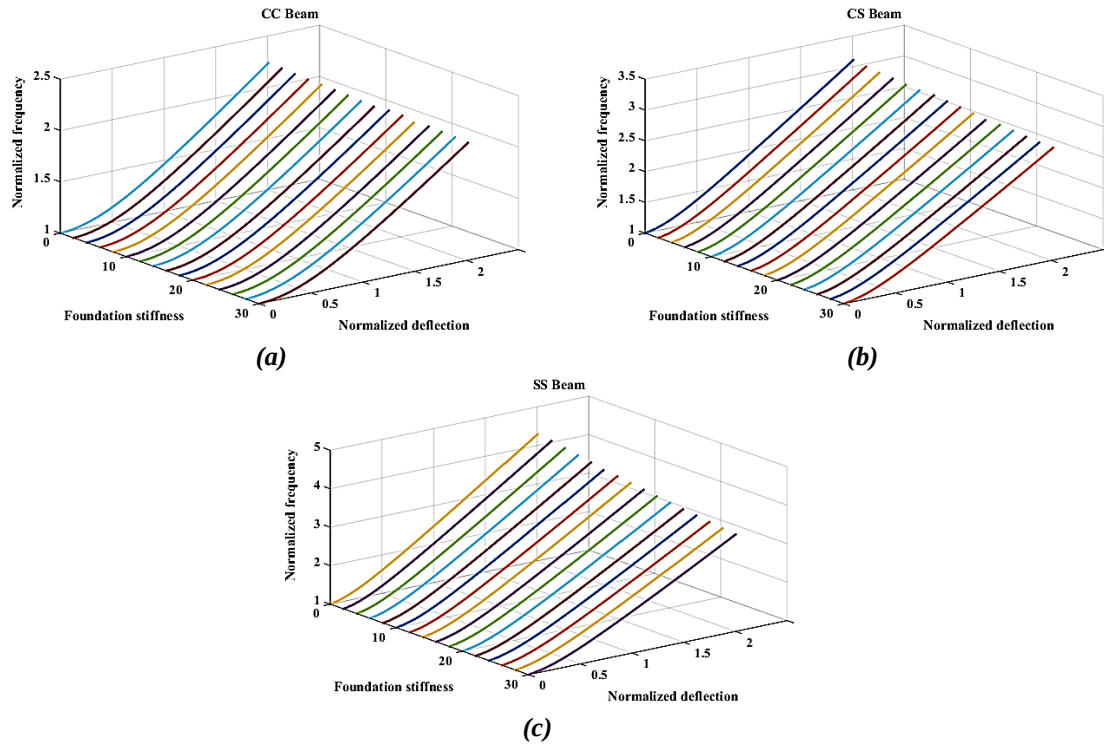


Fig. 5 - 1st BBCs for various combination of foundation stiffness values of Winkler foundation supported BFG (a) CC (b) CS and (c) SS beam.

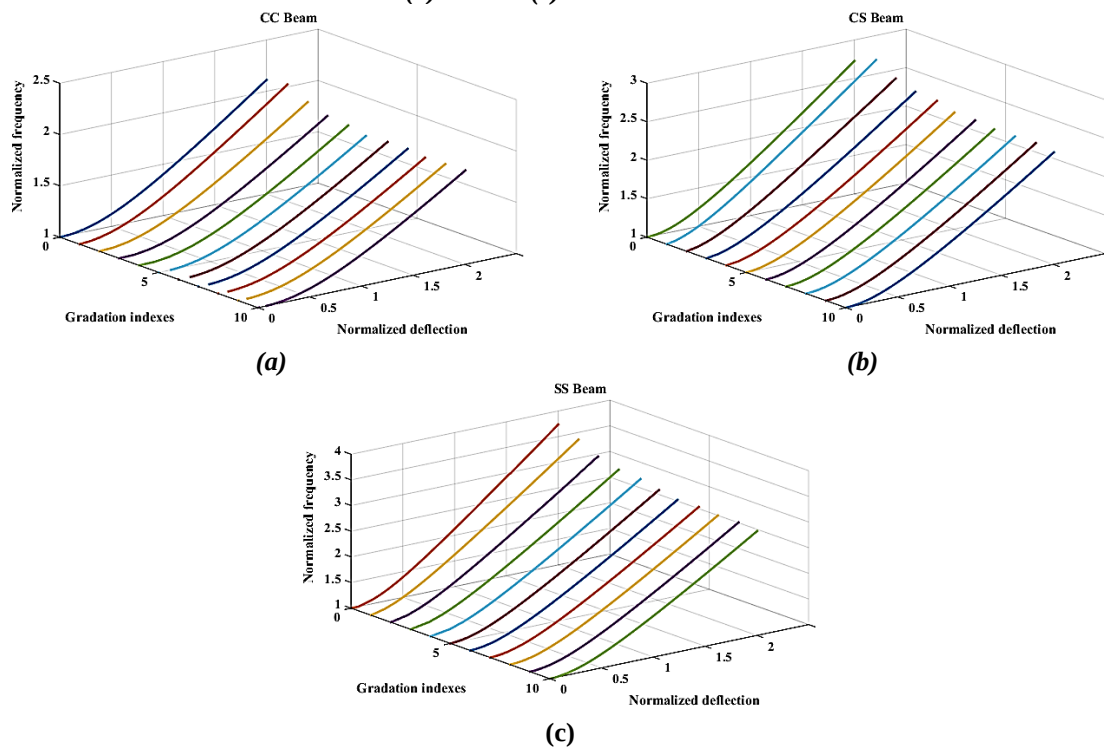


Fig. 6 - 1st BBCs for various combination of material gradation indexes of Kerr foundation supported BFG (a) CC (a) CS and (c) SS beam.

Fig. 6 shows the BBCs for the 1st vibrational mode corresponding to varied combinations of gradation indexes (n_x and n_z). Three set of plots are shown for distinct three BCs and each plot contains four BBCs for varying combination of n_x and n_z values, ranging from 0 to 10. The foundation stiffnesses are taken as constant at $k_u = k_l = 5$ kPa and $k_s = 5$ kN. The values of n_x and n_z could differ, but to keep brevity same values ($n_x = n_z$) are assigned. From the figure it is evident that across all instances slope of the BBC intensifies with gradation index value. The figure also shows that with the increase of gradation index the space between two adjacent BBCs is gradually decreasing.

BBCs corresponding to 2nd and 3rd modes of vibration are depicted in Fig. 7 for three distinct BCs. To produce the results, foundation stiffnesses are taken as $k_u = k_l = 5$ kPa and $k_s = 5$ kN with gradation indexes $n_x = n_z = 1$. While other higher order BBCs (past mode 3) are also obtainable but not presented to uphold the lucidity of the present plots. From Fig. 7, it is found that in CC beam 2nd mode linear natural frequency is around 2.7 times the linear fundamental frequency. It appears around 3 times and 6 times respectively in case of CS beam and SS beam. The 2nd mode and 3rd mode BBCs for SS beam are appearing close to each other in comparison to the same for CS beam and SS beam, while for CC beam, they appear with wide space. It is also evident that the slope of the BBCs for CC beam is maximum in comparison with the slope of BBCs in CS beam and SS beam, while the slope is minimum for SS beam.

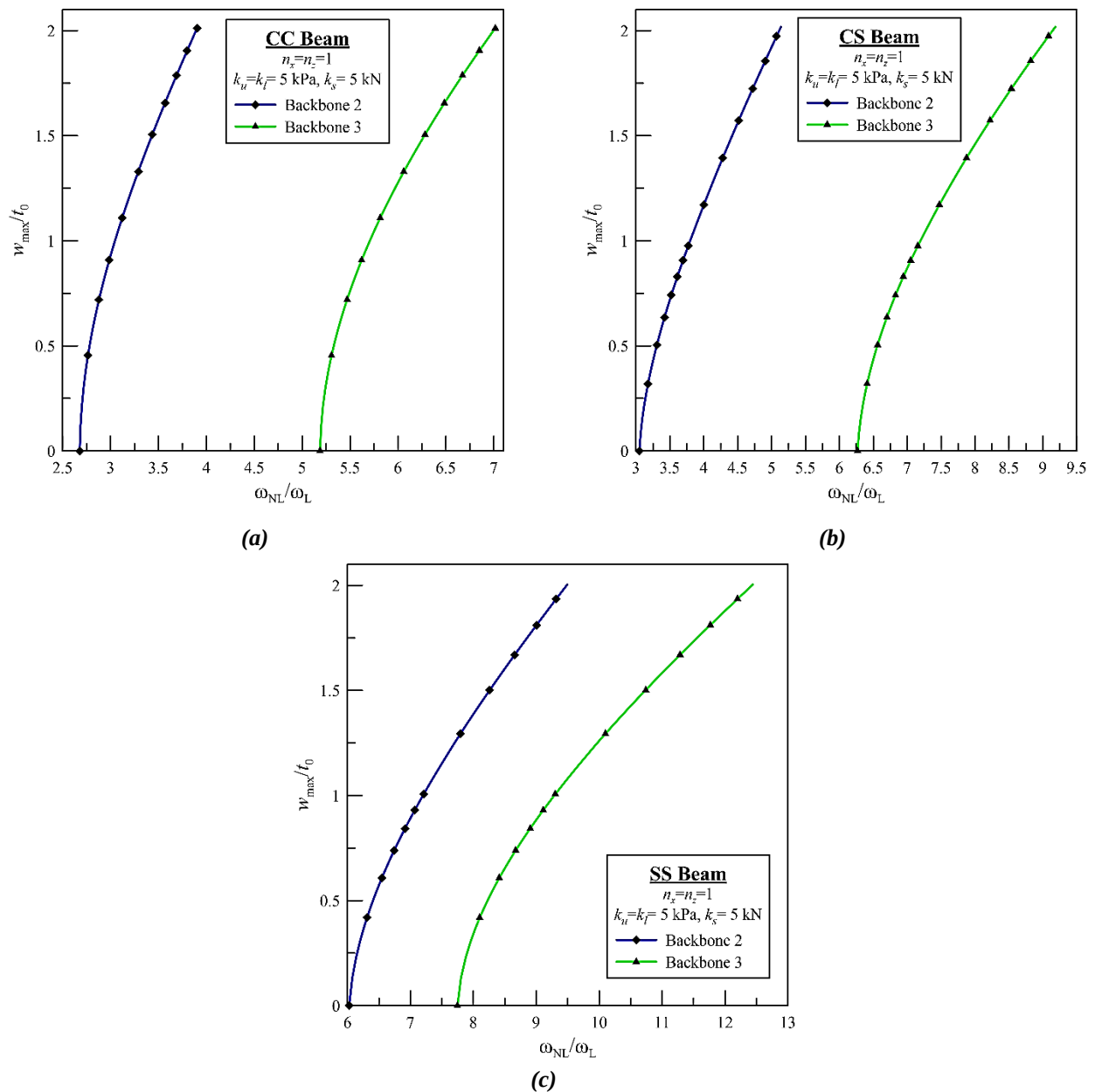


Fig. 7 - 2nd and 3rd BBCs of Kerr foundation supported BFG (a) CC (a) CS and (c) SS beam

Mode shapes (MSs) obtained from first three modes of vibration are presented in a normalized plane are illustrated in Fig. 8 for three dissimilar BCs. For each mode, two separate MSs are depicted: (1) linear MS, represented by continuous line and corresponding to the unloaded condition ($w_{\max}/h = 0$), while, (2) nonlinear MS, represented by dashed line and corresponding to loaded condition ($w_{\max}/h = 2.0$). To generate the results, foundation stiffnesses are taken as $k_u = k_l = 5$ kPa and $k_s = 5$ kN and both the gradation indexes are considered as $n_x = n_z = 1$. A clear gap between the linear and nonlinear MSs are evident in the figure. It is also observed that the difference between the two sets of MSs are more discernible in case of the asymmetric BC (CS). In the CS configuration, the clamped end restricts both translation and rotation, while the simply supported end allows rotation. This asymmetry results in a non-uniform curvature profile under loading, which is the possible reason for resulting greater accumulation of nonlinear deformation compared to symmetric BCs.

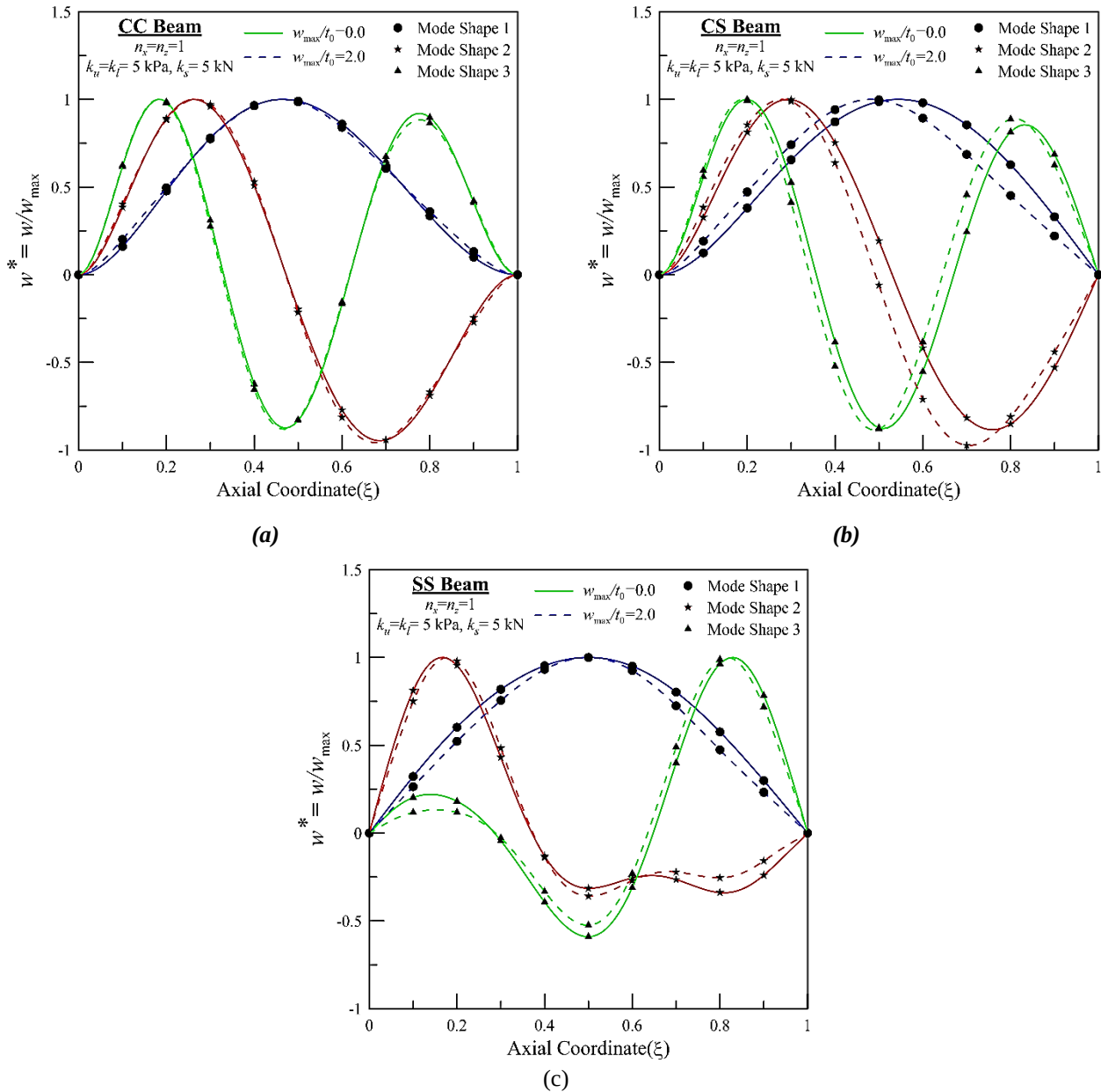


Fig. 8 - Linear and nonlinear first three MSs of Kerr foundation supported BFG (a) CC beam (a) CS beam and (c) SS beam

4 Conclusion

A thick BFG beam resting on Winkler, Pasternak or Kerr foundation under significant loading conditions is analysed in the present investigation to detect the large amplitude free vibration behaviour. In BFG beam, both the length and thickness

directional gradation of material properties are performed following power law distribution function. The determination of large amplitude vibrational response is carried out by performing a free vibration study on a largely and statically pre-deformed beam configuration, which involves two distinct but interconnected part, one is static while another is dynamics. Both of these, utilizes the variational approach of energy method to deduce the differential equation sets, whereas, the numerical form of direct substitution technique is employed for solving the static part of problem. The dynamic aspects of the problem are addressed using readily available static displacement field as an eigen value problem. The effectiveness of the adopted methodology is justified by conducting a validation study, where the results produced from the current methodology are tested with the established results. Fresh sets of findings are also demonstrated for several variations of system components to examine the influences of BFG material and various categories of elastic supports on the loaded natural frequencies.

The major observations obtained from presented results can be potted as follow,

- i. For particular BC, in Kerr foundation, natural frequency is maximum for homogeneous beam ($n_x = n_z = 0$) and minimum for maximum value of material gradation indexes ($n_x = n_z = 4$).
- ii. For all possible combination of BFG beam ($n_x = n_z = 1$), natural frequency rises with the rise of foundation stiffness.
- iii. For particular value of foundation stiffness in BFG beam ($n_x = n_z = 1$), natural frequency is maximum for Pasternak foundation and minimum for Winkler foundation.
- iv. For particular BC and foundation stiffness, natural frequency is maximum for Pasternak foundation and minimum for Winkler foundation.
- v. In all possible combination of foundation supported BFG CC beam shows maximum natural frequency while SS beam shows minimum natural frequency.
- vi. For all possible combination, the BBC represents stiffening type nonlinear system response.
- vii. For all cases of elastic foundation and material gradation, with the increase of stiffness values and gradation index value the slope of the BBC rises.

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Appendix A.

The elements of $[K_{ij}]$ are:

$$\begin{aligned}
 [K_{ij}]_{j=1,nw}^{i=1,nw} &= \frac{bh}{2L^3} \int_0^1 \int_0^1 \left(\sum_{k=1}^{nw} d_k \frac{d\phi_k}{d\xi} \right)^2 \frac{d\phi_i}{d\xi} \frac{d\phi_j}{d\xi} E(\xi, \eta) d\xi d\eta + \frac{bh}{L^2} \int_0^1 \int_0^1 \left(\sum_{k=nw+1}^{nw+nu} d_k \frac{d\phi_{k-nw}}{d\xi} \right) \frac{d\phi_i}{d\xi} \frac{d\phi_j}{d\xi} E(\xi, \eta) d\xi d\eta \\
 &- \frac{bh^2}{L^2} \int_0^1 \int_0^1 \left(\sum_{k=nw+nu+1}^{nw+nu+nsi} d_k \frac{d\beta_{k-nw-nu}}{d\xi} \right)^2 \frac{d\phi_i}{d\xi} \frac{d\phi_j}{d\xi} \left(\eta - \frac{1}{2} \right) E(\xi, \eta) d\xi d\eta + \frac{bhk_{sh}}{L} \int_0^1 \int_0^1 \frac{d\phi_i}{d\xi} \frac{d\phi_j}{d\xi} G(\xi, \eta) d\xi d\eta + \left(\frac{k_l k_u}{k_l + k_u} \right) L \int_0^1 \phi_i \phi_j d\xi \\
 &+ \left(\frac{k_s k_u}{k_l + k_u} \right) \frac{1}{L} \int_0^1 \frac{d\phi_i}{d\xi} \frac{d\phi_j}{d\xi} d\xi \\
 [K_{ij}]_{j=nw+1,nw+nu}^{i=1,nw} &= \frac{bh}{2L^2} \int_0^1 \int_0^1 \left(\sum_{k=1}^{nw} d_k \frac{d\phi_k}{d\xi} \right) \frac{d\phi_i}{d\xi} \frac{d\alpha_{j-nw}}{d\xi} E(\xi, \eta) d\xi d\eta \\
 [K_{ij}]_{j=nw+nu+1,nw+nu+nsi}^{i=1,nw} &= -bhk_{sh} \int_0^1 \int_0^1 \frac{d\phi_i}{d\xi} \beta_{j-nw-nu} G(\xi, \eta) d\xi d\eta - \frac{bh^2}{2L^2} \int_0^1 \int_0^1 \left(\sum_{k=1}^{nw} d_k \frac{d\phi_k}{d\xi} \right) \frac{d\phi_i}{d\xi} \frac{d\beta_{j-nw-nu}}{d\xi} \left(\eta - \frac{1}{2} \right) E(\xi, \eta) d\xi d\eta \\
 [K_{ij}]_{j=1,nw}^{i=nw+1,nw+nu} &= 0
 \end{aligned}$$

$$\begin{aligned}
[K_{ij}]_{j=nw+1, nw+nu}^{i=nw+1, nw+nu} &= \frac{bh}{L} \int_0^1 \int_0^1 \frac{d\alpha_{i-nw}}{d\xi} \frac{d\alpha_{j-nw}}{d\xi} E(\xi, \eta) d\xi d\eta \\
[K_{ij}]_{j=nw+nu+1, nw+nu+nsi}^{i=nw+1, nw+nu} &= -\frac{bh^2}{L} \int_0^1 \int_0^1 \frac{d\alpha_{i-nw}}{d\xi} \frac{d\beta_{j-nw-nu}}{d\xi} \left(\eta - \frac{1}{2}\right) E(\xi, \eta) d\xi d\eta \\
[K_{ij}]_{j=1, nw}^{i=nw+nu+1, nw+nu+nsi} &= -bhk_{sh} \int_0^1 \int_0^1 \beta_{i-nw-nu} \frac{d\phi_j}{d\xi} G(\xi, \eta) d\xi d\eta \\
[K_{ij}]_{j=nw+1, nw+nu}^{i=nw+nu+1, nw+nu+nsi} &= -\frac{bh^2}{L} \int_0^1 \int_0^1 \frac{d\beta_{i-nw-nu}}{d\xi} \frac{d\alpha_{j-nw}}{d\xi} \left(\eta - \frac{1}{2}\right) E(\xi, \eta) d\xi d\eta \\
[K_{ij}]_{j=nw+nu+1, nw+nu+nsi}^{i=nw+nu+1, nw+nu+nsi} &= \frac{bh^3}{L} \int_0^1 \int_0^1 \frac{d\beta_{i-nw-nu}}{d\xi} \frac{d\beta_{j-nw-nu}}{d\xi} \left(\eta - \frac{1}{2}\right)^2 E(\xi, \eta) d\xi d\eta + bhLk_{sh} \int_0^1 \int_0^1 \beta_{i-nw-nu} \beta_{j-nw-nu} G(\xi, \eta) d\xi d\eta
\end{aligned}$$

The elements of $\{f_i\}$ are,

$$\begin{aligned}
\{f_i\}_{i=1, nw} &= qL \int_0^1 \phi_i d\xi \\
\{f_i\}_{i=nw+1, nw+nu} &= 0 \\
\{f_i\}_{i=nw+nu+1, nw+nu+nsi} &= 0
\end{aligned}$$

The elements of $[M_{ij}]$ are,

$$\begin{aligned}
[M_{ij}]_{j=1, nw}^{i=1, nw} &= Lbh \int_0^1 \int_0^1 \phi_i \phi_j \rho(\xi, \eta) d\xi d\eta \\
[M_{ij}]_{j=nw+1, nw+nu}^{i=nw+1, nw+nu} &= Lbh \int_0^1 \int_0^1 \alpha_{i-nw} \alpha_{j-nw} \rho(\xi, \eta) d\xi d\eta \\
[M_{ij}]_{j=nw+nu+1, nw+nu+nsi}^{i=nw+nu+1, nw+nu+nsi} &= Lbh^3 \int_0^1 \int_0^1 \beta_{i-nw-nu} \beta_{j-nw-nu} \left(\eta - \frac{1}{2}\right)^2 \rho(\xi, \eta) d\xi d\eta
\end{aligned}$$

Remaining elements of $[M_{ij}]$ are zero.

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