



Journal of Materials and Engineering Structures

Research Paper

Stochastic Stability Analysis of Columns with Randomly Elastic Joint Ends and Two-Dimensional Random Material Properties Using Monte Carlo Simulation

Nguyen Dang Diem^a, Dao Sy Dan^b, Ta Duy Hien^b, Dao Ngoc Tien^{c,*}

^a Campus in Ho Chi Minh City, University of Transport and Communications, Ho Chi Minh City, Viet Nam

^b University of Transport and Communications, Hanoi, Vietnam

^c Hanoi Architectural University, Km10 Nguyen Trai Street, Thanh Xuan, Hanoi, Vietnam

ARTICLE INFO

Article history :

Received : 29 October 2024

Revised : 19 March 2025

Accepted : 20 March 2025

Keywords:

SFEM

Monte Carlo simulation

Column

Random field

ABSTRACT

This paper develops a Monte Carlo simulation (MCs) to analyze the stochastic stability of a pin-ended column with elastic rotational springs at both ends. The randomness of input factors is considered, including the rotational stiffness of elastic couplings at both ends of the column as random variables following a normal distribution and the elastic modulus of the column as a two-dimensional (2D), stationary, homogeneous Gaussian random field. The spectral representation method is applied to represent the 2D random field and generate realizations of the elastic modulus. The influence of random factors such as the standard deviation of the 2D random field, the standard deviation of normal random variables, and correlation distance in each direction on the coefficient of variation (COV) of the critical load is analyzed in detail in this study. The results indicate a strong correlation between the COV of the critical load and both the standard deviation of the 2D random field and the correlation distance. The COV increases significantly with increasing standard deviation, particularly for larger correlation lengths. However, the influence of the variability of rotational stiffness is relatively minor, especially at larger correlation distances.

1 Introduction

When analyzing structures, input factors such as loads, geometric dimensions, boundary conditions, or material properties can be random or uncertain [1]. These random or uncertain factors are mathematically represented as random variables or random fields [2]. The finite element method (FEM) is a popular numerical method used to analyze structures with complex geometries, various boundary conditions, and a wide range of materials [3, 4]. However, due to its deterministic nature, the finite element method cannot describe the randomness of input factors or analyze the random response of

* Corresponding author. Tel.: +084 0988185072.

E-mail address: tiendn@hau.edu.vn

structures. When input parameters are uncertain, the structural response is also uncertain. By merging probability theory and the standard FEM, the stochastic finite element method (SFEM) provides a powerful tool for analyzing systems with random inputs [5]. The SFEM is widely applied in engineering and other scientific fields such as materials science [6, 7], biomechanics [8, 9], and geotechnical engineering [10-12].

To analyze the random response of structures with random input parameters using the SFEM, two primary steps are typically adopted [13]. The first step involves discretizing the random field into random variables within the finite element framework. Various techniques have been developed and applied, including point discretization methods, mean-based methods, and perturbation expansion methods [5]. The second step in the SFEM is to propagate the randomness into the structure and analyze the random response. Methods developed and applied for this purpose include the Neumann expansion method [14], spectral finite element method [14, 15], perturbation method [16], and Monte Carlo simulation (MCs) [17].

The SFEM based on MCs is a method that combines standard finite element analysis with the simulation of random fields associated with input parameters. MCs is a group of numerical methods for structural random analysis based on statistical principles [18, 19]. The first step in the simulation process is to generate realizations of the stochastic field. Several methods are developed to generate realizations of stochastic fields, such as the spectral representation method [20, 21], the Fourier transform method [22, 23], and direct Monte Carlo simulation [24]. Based on the generated input dataset combined with standard FEM, the structural response is obtained, resulting in a set of random responses of the structure. Statistical processing of the obtained random response dataset provides statistical characteristics of the structural response, such as mean, variance, standard deviation, and coefficient of variation. The SFEM based on MCs is a direct method suitable for stochastic analysis problems [13]. However, due to its statistical nature, this method requires generating many realizations of the input random field, leading to an enormous computational cost and long computation time. Nevertheless, techniques have been developed to reduce the computational time to overcome this drawback [25, 26].

The stability analysis of columns with random axial loads or random material properties has been a subject of significant research and development. S. Anantha Ramu and R. Ganesan [27] analyzed the stability of columns with random modulus of elasticity and uniformly distributed random loads along the column length. Perturbation techniques and weighted residual methods were employed to analyze and represent the stochastic field. The statistical characteristics of the buckling load, such as mean and variance, were obtained. R. Ganesan and VK Kowda [28] developed a SFEM to analyze the stability of columns made of composite materials with unidirectional fibers. The modulus of elasticity and cross-sectional width were selected as random variables varying along the column length. The authors used the perturbation method for random stability analysis in the case of rectangular and thin-walled cross-sections. DT Hang et al. [29] developed a SFEM based on the perturbation techniques combined with the random field point representation technique to analyze the stability of columns with linearly varying cross-sections along the column length. Similarly, the stability analysis of columns with a first-order varying cross-section along the column length and random modulus of elasticity was conducted. T.D. Hien [30] used the weighted residual method to represent the random field of the column varying along the length. The first-order perturbation technique combined with the SFEM was used for the stability analysis. The stochastic modulus of elasticity field was represented using the weighted residual method. The paper obtained the first-order statistical characteristics of the critical load value.

In addition to the commonly considered random factors such as material properties, cross-sectional dimensions, or loads, research has been developed to consider the initial curvature of the column as a random field [31]. The authors analyzed the stability of columns with random curvature and concluded that open cross-sections are more susceptible to torsion, resulting in a lower compressive capacity compared to closed cross-sections. The stochastic stability problem of columns was extended to composite materials with unidirectional fibers. Studies have focused on off-axis unidirectional fiber composites. For example, VK Kowda [32] analyzed the static and dynamic stability of columns made of composite materials with solid and thin-walled cross-sections, considering uncertainties in material properties, geometric dimensions, boundary conditions, and loads. For this type of material, Nguyen, Hoang X et al. [33] used the spectral representation technique to model the random material properties of columns made of laminated composite beams with unidirectional fibers. For stochastic analysis, the authors used the MCs method combined with the isoparametric finite element method to analyze the column's stochastic stability. The probabilistic model of random fields was also developed and extended to the case of non-homogeneous random fields and non-Gaussian [34] to analyze single-layer composite beams with random modulus of elasticity and shear modulus varying along the length.

Although numerous studies have investigated the stochastic stability of columns with a one-dimensional random elastic modulus and fixed boundary conditions, limited research has simultaneously considered the randomness of the elastic

modulus in two spatial dimensions and the effects of elastic boundary conditions. Most prior studies model the variation of the elastic modulus only along the column length, neglecting its variation across the cross-section, while the influence of randomly varying rotational stiffness at the boundaries on the critical load remains insufficiently explored. To address these limitations, this study develops a stochastic stability model for columns with elastic boundary conditions and randomly varying rotational stiffness, incorporating a two-dimensional random elastic modulus using the Monte Carlo simulation (MCs) method. The spectral representation method is employed to generate realizations of the two-dimensional random field and integrate them into the stochastic finite element framework. Through a parametric analysis, this study evaluates the influence of correlation length, the standard deviation of the elastic modulus, and the variability of rotational stiffness on the coefficient of variation (COV) of the critical load. The results indicate that the standard deviation of the elastic modulus significantly affects the COV of the critical load, while the influence of randomly varying rotational stiffness diminishes as the correlation length increases. These findings provide deeper insights into the impact of stochastic factors on the stability of column structures, contributing to the improvement of computational models for structural mechanics problems involving uncertainties.

2 Stochastic stability analysis of a column with 2D random material properties using MCs

A centrally loaded column with a rectangular cross-section and length L is considered. The column's modulus of elasticity is assumed to be a 2D random field. Both ends of the column are subject to rotational restraints provided by springs with random rotational spring constants.

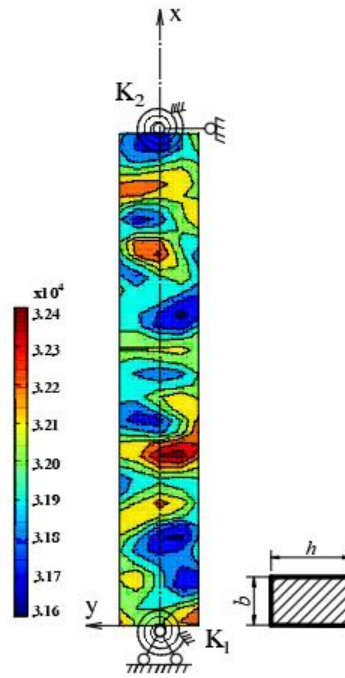


Fig. 1 – A column with a rotational elastically restrained end and 2D varying random material properties

Assuming a 2D homogeneous Gaussian random field for the column's modulus of elasticity, as defined by Equation (1), with E_0 representing the mean value and $f(x,y)$ being a zero-mean, stationary, Gaussian random field, the 2D variability of the material property is incorporated into the analysis. Additionally, the rotational elastic restraints at both ends are modeled as random variables following a normal distribution described by Equations (2) and (3) with their mean values defined by K_{01} and K_{02} .

$$E = E_0 [1 + f(x,y)] \quad (1)$$

$$K_1 = K_{01} [1 + r_1] \quad (2)$$

$$K_2 = K_{02} [1 + r_2] \quad (3)$$

Using spectral representation [20, 21], $f(x,y)$ can be decomposed into a sum of harmonic functions characterized by distinct frequencies and phases. This decomposition allows for interpreting the random field as a superposition of numerous simple sinusoidal waves. Each sinusoidal wave corresponds to a specific frequency component within the random field.

$$f(x,y) = \sqrt{2} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \left\{ \begin{aligned} &A_{n_1 n_2} \cos \left[\kappa_{1n_1} x + \kappa_{2n_2} y + \Phi_{n_1 n_2}^{(1)} \right] \\ &+ B_{n_1 n_2} \cos \left[\kappa_{1n_1} x - \kappa_{2n_2} y + \Phi_{n_1 n_2}^{(2)} \right] \end{aligned} \right\} \quad (4)$$

In Equation (4), $A_{n_1 n_2}$ and $B_{n_1 n_2}$ denote the random amplitudes of the n -th harmonic function. These quantities are defined by Equation (5), where S_{ff} represents the power spectral density of the 2D random field. $\Phi_{n_1 n_2}^{(1)}$ and $\Phi_{n_1 n_2}^{(2)}$ are uniformly distributed random variables on the interval $[0, 2\pi]$, representing random phases and assumed to be independent.

$$\begin{aligned} A_{n_1 n_2} &= \sqrt{2S_{ff}(\kappa_{1n_1}, \kappa_{2n_2}) \Delta\kappa_1 \Delta\kappa_2} \\ B_{n_1 n_2} &= \sqrt{2S_{ff}(\kappa_{1n_1}, -\kappa_{2n_2}) \Delta\kappa_1 \Delta\kappa_2} \end{aligned} \quad (5)$$

The frequency of the n -th harmonic component is determined by Equation (6), where κ_{1u} and κ_{2u} represent the upper cut-off limits of wave numbers.

$$\kappa_{1n_1} = n_1 \frac{\kappa_{1u}}{N_1}; \quad \kappa_{2n_2} = n_2 \frac{\kappa_{2u}}{N_2} \quad (6)$$

The column is divided into N_e elements of length L_e . According to Equation (7), the element nodal displacement vector $\{\delta\}_i^e$ and shape function matrix $[N]$ approximate the element displacement $\{v\}_i$.

$$\{v\}_i = [N] \{\delta\}_i^e \quad (7)$$

Equation (8) presents the column's governing stability equation, formulated using finite element discretization. Here, K , K_p , and U represent the global stiffness matrix, geometric stiffness matrix, and global displacement vector, respectively, with U corresponding to the buckling mode shape.

$$[K - \lambda K_p]U = 0 \quad (8)$$

The eigenvalues of Equation (8) represent the critical loads. The minimum eigenvalue, denoted by λ , corresponds to the lowest critical load, also known as the buckling load. K is assembled from the element stiffness matrices and the elastic joint stiffness matrix. The element stiffness matrix is defined by Equation (10). The elastic joint stiffness matrix $[K]_s$ is diagonal, the same size as the global stiffness matrix. The diagonal elements corresponding to the rotational degrees of freedom at both ends of the column are equal to the rotational spring stiffness. All other diagonal elements are zero.

$$K = \sum_{e=1}^{N_e} [K]_e + [K]_s \quad (9)$$

$$[K]_e = \int_{V_e} y_e^2 E [N'']^T [N''] dV_e \quad (10)$$

By substituting Equations (1) and (4) into Equation (10) and transforming the result to the local element coordinate system, we derive Equation (11), with Δ_e denoting the distance from the beam end to the e -th element.

$$\begin{aligned} [K]_e &= \underbrace{\int_{V_e} y_e^2 E_0 [N'']^T [N''] dV_e}_{[K_0]_e} \\ &+ \underbrace{\int_{V_e} y_e^2 E_0 \sqrt{2} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \left\{ \begin{aligned} &A_{n_1 n_2} \cos \left[\kappa_{1n_1} (x_e + \Delta_e) + \kappa_{2n_2} y_e + \Phi_{n_1 n_2}^{(1)} \right] \\ &+ B_{n_1 n_2} \cos \left[\kappa_{1n_1} (x_e + \Delta_e) - \kappa_{2n_2} y_e + \Phi_{n_1 n_2}^{(2)} \right] \end{aligned} \right\} [N'']^T [N''] dV_e}_{[\Delta K]_e} \end{aligned} \quad (11)$$

The element stiffness matrix is modeled as a combination of a deterministic and a stochastic component. Equation (12) defines the deterministic part, and Equation (13) governs the stochastic part. In this study, numerical integration is utilized to quantify the stochastic component of the element stiffness matrix.

$$[K_0]_e = \int_{V_e} y_e^2 E_0 [N'']^T [N''] dV = \frac{E_0 I}{L_e^3} \begin{bmatrix} 12 & 6L_e & -12 & 6L_e \\ & 4L_e^2 & -6L_e & 2L_e^2 \\ & & 12 & -6L_e \\ sym & & & 4L_e^2 \end{bmatrix} \quad (12)$$

$$[\Delta K]_e = \int_{V_e} y_e^2 E_0 \sqrt{2} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \left\{ \begin{aligned} & A_{n_1 n_2} \cos \left[\kappa_{1n_1} (x_e + \Delta_e) + \kappa_{2n_2} y_e + \Phi_{n_1 n_2}^{(1)} \right] \\ & + B_{n_1 n_2} \cos \left[\kappa_{1n_1} (x_e + \Delta_e) - \kappa_{2n_2} y_e + \Phi_{n_1 n_2}^{(2)} \right] \end{aligned} \right\} [N'']^T [N''] dV_e \quad (13)$$

Equation (14) is obtained by substituting Equations (12) and (13) into Equation (9).

$$K = \sum_{e=1}^{N_e} [K_0]_e + \sum_{e=1}^{N_e} [\Delta K]_e + [K]_s \quad (14)$$

The geometric stiffness matrix K_p is formed by summing up the element geometric stiffness matrices, as expressed in Equation (15). The element geometric stiffness matrix is defined as in (16).

$$K_p = \sum_{e=1}^{N_e} [K_p]_e \quad (15)$$

$$[K_p]_e = \int_0^{L_e} [N']^T [N'] dx = \begin{bmatrix} \frac{6}{5L_e} & \frac{1}{10} & -\frac{6}{5L_e} & \frac{1}{10} \\ & \frac{2L_e}{15} & -\frac{1}{10} & -\frac{L_e}{30} \\ & & \frac{6}{5L_e} & -\frac{1}{10} \\ sym & & & \frac{2L_e}{15} \end{bmatrix} \quad (16)$$

The stability equation for a column with random rotational elastic restraint and 2D random elastic modulus is derived by substituting Eqs. (14) and (16) into Equation (8), resulting in Equation (17).

$$\left[\left(\sum_{e=1}^{N_e} [K_0]_e + \sum_{e=1}^{N_e} [\Delta K]_e + [K]_s \right) - \lambda \left(\sum_{e=1}^{N_e} [K_p]_e \right) \right] \{U\} = 0 \quad (17)$$

The existence of a non-trivial solution to Equation (17) requires the following condition:

$$Det \left[\left(\sum_{e=1}^{N_e} [K_0]_e + \sum_{e=1}^{N_e} [\Delta K]_e + [K]_s \right) - \lambda \left(\sum_{e=1}^{N_e} [K_p]_e \right) \right] = 0 \quad (18)$$

Using Equation (4), N sample functions of the 2D random field of the elastic modulus are generated, along with N sets of standard normal variates r_1 and r_2 with means of zero and standard deviations of σ_1 and σ_2 , respectively. Equation (19) is then employed to determine the corresponding N critical loads. The resulting dataset provides a distribution of critical loads based on the random field variations.

$$Det \left[\left(\sum_{e=1}^{N_e} [K_0]_e^i + \sum_{e=1}^{N_e} [\Delta K]_e^i + [K]_s^i \right) - \lambda_i \left(\sum_{e=1}^{N_e} [K_p]_e^i \right) \right] = 0 \quad (19)$$

Statistical analysis of the set of N critical load values yields the statistical descriptors of the critical load, including the mean, variance, and coefficient of variation, as defined in Equations (20), (21), and (22), respectively.

$$E[\lambda] = \frac{1}{N} \sum_{i=1}^N \lambda_i \quad (20)$$

$$\text{Var}[\lambda] = \frac{1}{N-1} \sum_{i=1}^N (\lambda_i - E[\lambda])^2 \quad (21)$$

$$\text{COV} = \frac{\sqrt{\text{Var}[\lambda]}}{E[\lambda]} \quad (22)$$

3 Results and discussion

Consider a centrally compressed column with elastically hinged ends. The column has a length $L = 3.5 \text{ m}$, a rectangular cross-section with width $b = 0.4 \text{ m}$ and height $h = 0.4 \text{ m}$. The rotational spring stiffness of the elastic hinges varies randomly following a normal distribution with a mean value $K_{01} = K_{02} = K = 19.5 \times 10^5 \text{ N.m/rad}$. The random variables r_1 and r_2 are characterized by standard deviations of σ_1 and σ_2 . The Young's modulus varies randomly in two dimensions with a mean value $E_0 = 32.10^3 \text{ MPa}$. The 2D Gaussian random field has an autocorrelation function as defined in Equation (23) with a standard deviation of $\sigma_f = 0.05$ and correlation lengths d_x and d_y in the x and y directions, respectively.

$$R_{ff}(x_2 - x_1, y_2 - y_1) = \sigma_f^2 \exp \left[-\left(\frac{x_2 - x_1}{d_x} \right)^2 - \left(\frac{y_2 - y_1}{d_y} \right)^2 \right] \quad (23)$$

A numerical study was conducted on a column with random material properties and random rotational spring stiffness at both ends. $N = 10,000$ samples of a two-dimensional random elastic field were generated using spectral representation. Figure 3 illustrates the sample function of a two-dimensional random field simulated using spectral representation method for various cases of correlation lengths between columns in space.

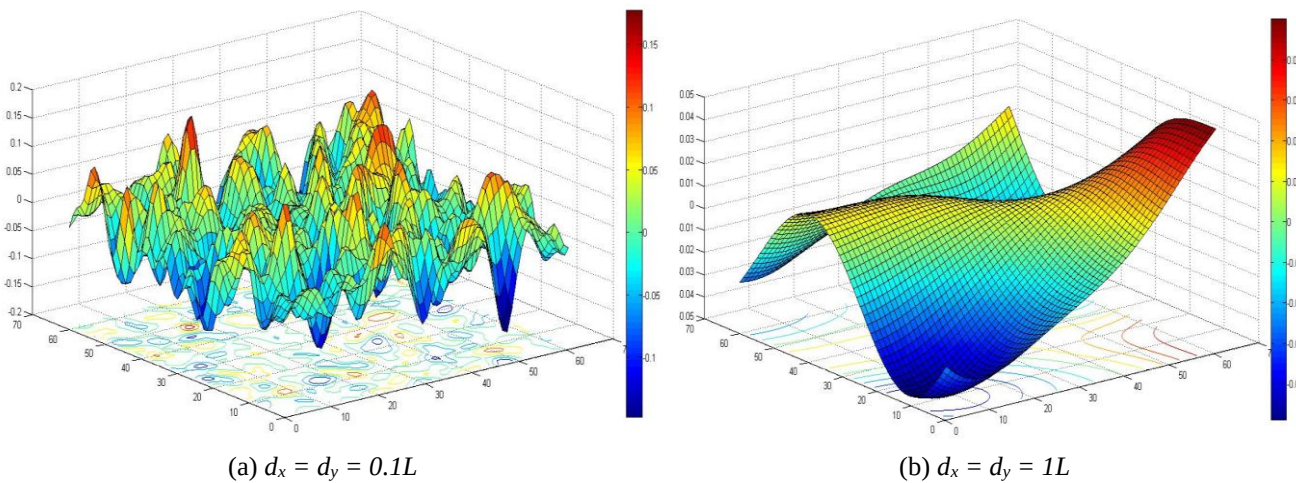


Fig. 2 – Realization of a 2D random field

Create a sample of size $N = 10,000$ from each of the normally distributed random variables r_1 and r_2 . As described in Equation (19), the standard FEM was employed to determine each sample's critical buckling load. The statistical properties of the critical buckling load were investigated for different correlation lengths ($d_x = d_y = 0.005L$, $d_x = d_y = 0.1L$, $d_x = d_y = L$, $d_x = d_y = 15L$) and standard deviations ($\sigma_f = 0.05$, $\sigma_1 = \sigma_2 = 0.05$). The results are illustrated in Figure 3 the critical force dataset was subjected to statistical analysis to determine mean, variance, standard deviation, and coefficient of variation.

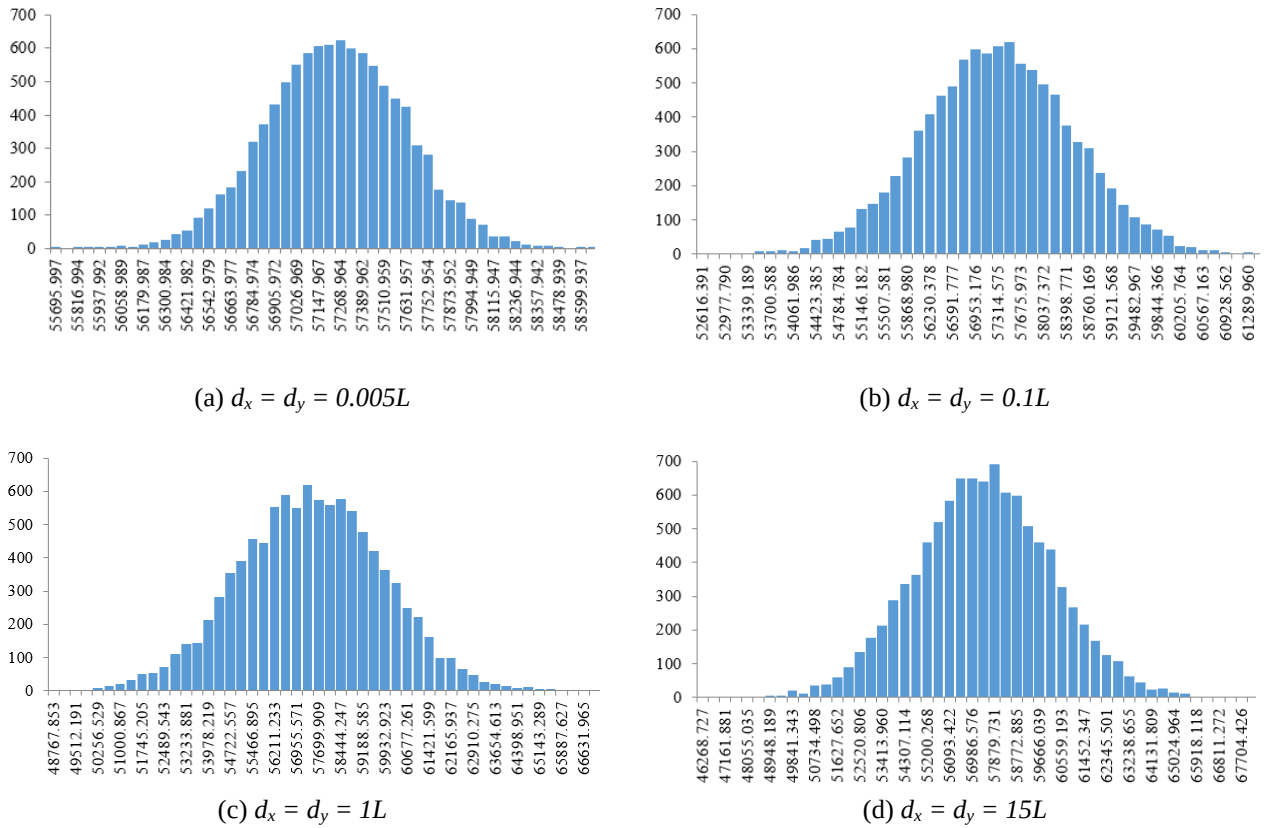


Fig. 3 – Frequency distribution of critical load

The accuracy and computational efficiency of MCs in stochastic stability analysis depend significantly on its convergence behavior. As a statistical method, MCs exhibits a convergence rate proportional to $1/\sqrt{N}$, where N is the number of realizations. This inherently slow convergence necessitates a sufficiently large sample size to ensure statistical reliability in estimating the coefficient of variation (COV) of the critical load. In this study, $N=10,000$ realizations were employed to achieve stable statistical characteristics. The presence of a two-dimensional (2D) random elastic modulus field further increases the computational complexity, as spatial correlations influence the variability of the system response. The spectral representation method used for random field generation introduces additional considerations for convergence, as the number of retained spectral components affects the accuracy of the simulated field. Moreover, the correlation length of the random field plays a crucial role in convergence behavior. Shorter correlation lengths lead to localized variations, requiring finer discretization and potentially more realizations, whereas longer correlation lengths result in smoother variations, reducing the required sample size. Despite the robustness of MCs, its computational cost remains a major concern. Variance reduction techniques, such as Latin Hypercube Sampling (LHS) [35] or quasi-Monte Carlo methods [36], could be employed in future studies to improve convergence efficiency. The results of this study indicate that beyond a certain threshold of N , additional samples provide diminishing returns in improving statistical accuracy, suggesting the potential for adaptive sampling strategies to optimize computational effort in stochastic structural analysis.

Figure 4 presents the results of the analysis on the influence of correlation length in the two-dimensional space on the COV of the critical force. Figure 3a corresponds to the case where $d_x = d_y = d$, with d varying from $0.005L$ to $50L$. Figure 3b shows the analysis of the COV of the critical force as a function of d_x and d_y , with d_x and d_y varying from $0.005L$ to $50L$. It can be observed that the COV tends to increase as the correlation length in the spatial directions increases. When the correlation length of the elastic modulus is small, nearby points are correlated, resulting in small variations in the elastic modulus around the mean value and, consequently, small variations in the critical force around its mean value. Therefore, the COV of the critical force is small. Conversely, when the correlation length of the elastic modulus is large, nearby points have little or no correlation in their elastic moduli, leading to large fluctuations in the critical force values around the mean. As a result, the COV of the critical force is large, corresponding to a large correlation length.

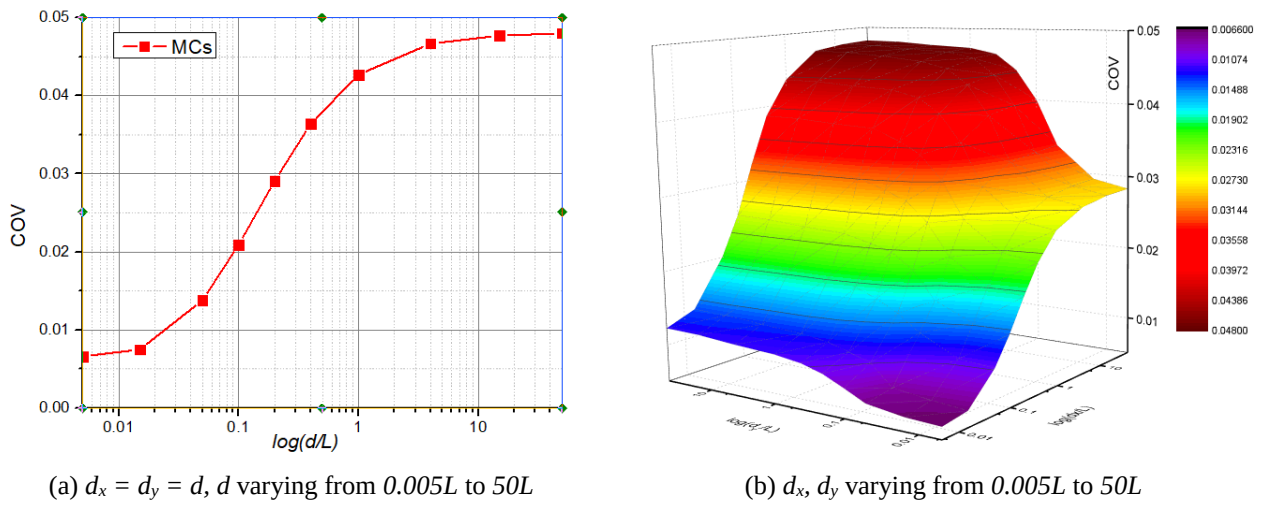


Fig. 4 – The influence of dx and dy on the COV of the critical force

The study investigated the influence of random input parameters, specifically the standard deviation of the 2D random field, on the COV of the critical force while maintaining a constant rotational stiffness standard deviation ($\sigma_1 = \sigma_2 = 0.05$). The results are presented in Figure 5a. Similarly, the effect of the standard deviation of the rotational stiffness on the critical force's COV was analyzed with a constant standard deviation of the 2D random field ($\sigma_f = 0.05$). The results are shown in Figure 5b. It is observed that the standard deviation of the 2D random field has a significant impact on the COV of the critical force. The COV tends to increase with increasing standard deviation of the random field, especially for considerable correlation lengths. The influence of the rotational stiffness variability on the critical force's COV is relatively tiny for large correlation lengths. However, for considerable correlation lengths, the effect of the elastic modulus variability on the critical force's COV is more pronounced.

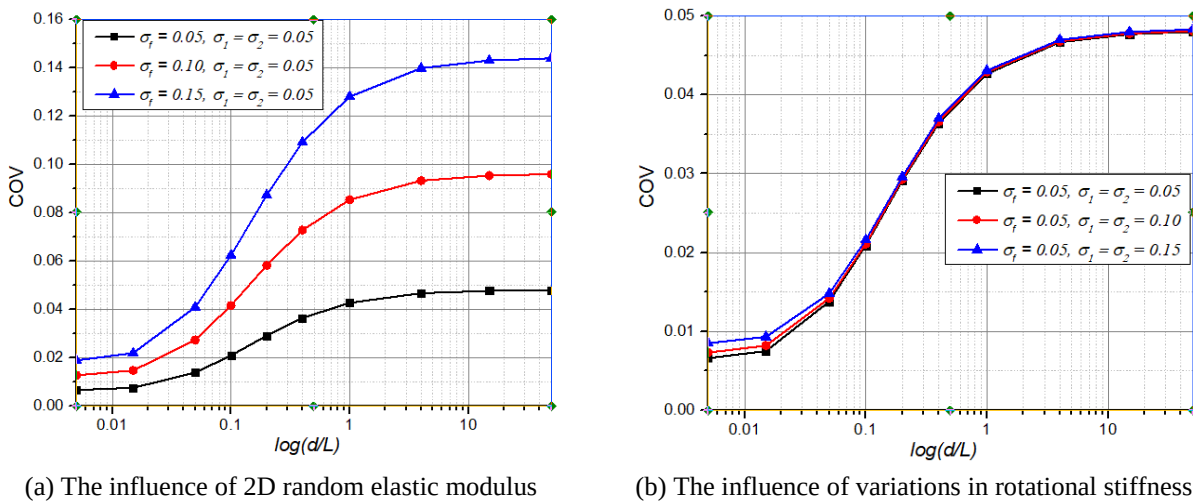


Fig. 5 – Influence of random input parameters

The influence of the rotational stiffness of flexible couplings on the COV of the critical load is illustrated in Figure 6. Three cases of expected values of rotational stiffness were investigated: $K_{01} = K_{02} = 0$, $K_{01} = K_{02} = K$, and $K_{01} = K_{02} = 5K$. In the case of $K_{01} = K_{02} = 0$, corresponding to the case of both ends of the column being freely rotatable, the COV of the critical load has the highest value compared to the other cases. The COV of the critical load tends to decrease as the expected value of rotational stiffness increases. This is clearly shown for large correlation distances.

The analysis results indicate that the randomness of the elastic modulus significantly affects the variability of the critical load, thereby directly impacting the structural reliability. As the standard deviation of the random field increases, the coefficient of variation (COV) of the critical load also increases, highlighting a greater level of uncertainty in the column's load-carrying capacity. This implies that structural performance can be significantly influenced by material variability, particularly in designs requiring high safety factors. Furthermore, the correlation length plays a crucial role: for small correlation lengths, material variations occur at a local scale, potentially creating weak regions that reduce overall stiffness and increase the risk of local instability. In contrast, for larger correlation lengths, the elastic modulus varies gradually in space, mitigating the impact of localized weaknesses while still influencing the overall structural behavior.

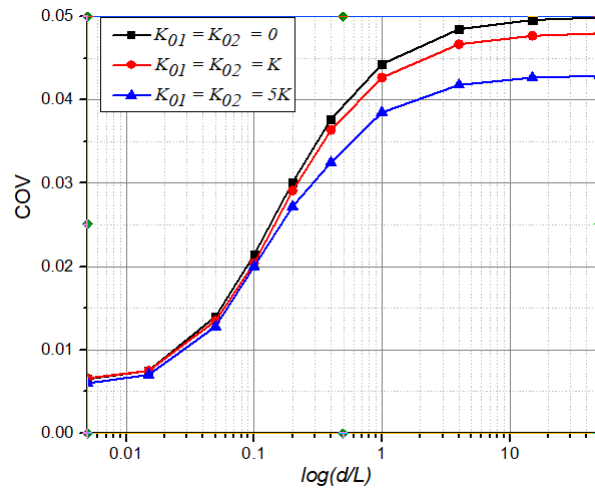


Fig. 6 – The influence of rotational stiffness on the COV of the critical load

These findings emphasize the necessity of fully considering material uncertainties in structural reliability analysis. Traditional deterministic models may fail to accurately capture the effects of material variability, especially in slender structures that are highly sensitive to random fluctuations. Therefore, the application of stochastic analysis methods, such as the stochastic finite element method (SFEM) or Monte Carlo simulation (MCs), is essential for more accurate predictions of structural load-bearing capacity. These findings are not only relevant to the stability analysis of columns but can also be extended to more complex structural systems, where material randomness plays a dominant role in reliability assessment.

4 Conclusions

This study successfully employed Monte Carlo simulations to analyze the stochastic stability of a pin-ended column with random rotational stiffness of elastic couplings and 2D random material properties. The rotational stiffness of the elastic couplings is assumed to be a random variable following a normal distribution, and the elastic modulus is a 2D stationary, homogeneous Gaussian random field. The influence of random input factors on the COV of the critical load was also analyzed in detail. The impact of the standard deviation of the 2D random field of elastic modulus on the COV is significant, with the COV tending to increase as the standard deviation of the random field increases. Meanwhile, the influence of the standard deviation of the random variable of rotational stiffness of elastic couplings on the COV is relatively small. Furthermore, an increase in the correlation distance of the 2D random field leads to an increase in the COV. Still, when the correlation distance is sufficiently large, the increase in the COV is negligible as the correlation distance continues to increase. The results of this study allow for the prediction of the stochastic response of the column given the random input factors.

REFERENCES

- [1]- G.I. Schuëller, Developments in stochastic structural mechanics, *Archive of Applied Mechanics*, 75 (2006), 755-73. <https://doi.org/10.1007/s00419-006-0067-z>.
- [2]- T.B. Wing Kam Liu, A. Mani, Random field finite elements, *International Journal for Numerical Methods in Engineering*, 23 (1986), 1831-45. <https://doi.org/10.1002/nme.1620231004>.
- [3]- D.L. Logan, *A first course in the finite element method*, Thomson, 2011.

- [4]- P.T.B. Lien, Free vibration of porous functionally graded sandwich beams on elastic foundation based on trigonometric shear deformation theory, *Transport and Communications Science Journal*, 74 (2023), 946-61. <https://doi.org/10.47869/tcsj.74.8.8>
- [5]- G. Stefanou, The stochastic finite element method: Past, present and future, *Computer Methods in Applied Mechanics and Engineering*, 198 (2009), 1031-51. <https://doi.org/10.1016/j.cma.2008.11.007>.
- [6]- Y. Yu, J. Ye, Y. Wang, B. Zhang, G. Qi, A mesoscale ultrasonic attenuation finite element model of composites with random-distributed voids, *Composites Science and Technology*, 89 (2013), 44-51. <https://doi.org/10.1016/j.compscitech.2013.09.006>.
- [7]- O.E. Sotomayor, H.V. Tippur, Role of cell regularity and relative density on elasto-plastic compression response of random honeycombs generated using Voronoi diagrams, *International Journal of Solids and Structures*, 51 (2014), 3776-86. <https://doi.org/10.1016/j.ijsolstr.2014.07.009>.
- [8]- Q.W.D.S. Strait, P.C. Dechow, C.F. Ross, B.G. Richmond, M.A. Spencer, B.A. Patel, Modeling elastic properties in finite - element analysis: how much precision is needed to produce an accurate model? *The Anatomical Record Part A: Discoveries in Molecular, Cellular, and Evolutionary Biology*, 283 (2005), 275-87. <https://doi.org/10.1002/ar.a.20172>.
- [9]- C.F. Ross, M.A. Berthaume, P.C. Dechow, J. Iriarte-Diaz, L.B. Porro, B.G. Richmond, M. Spencer, D. Strait, In vivo bone strain and finite-element modeling of the craniofacial haft in catarrhine primates. *Journal of Anatomy*, 218 (2011) 112-41. <https://doi.org/10.1111/j.1469-7580.2010.01322.x>.
- [10]- D.V. Griffiths, J. Paiboon, J. Huang, G.A. Fenton, Homogenization of geomaterials containing voids by random fields and finite elements. *International Journal of Solids and Structures*, 49 (2012) 2006-14. <https://doi.org/10.1016/j.ijsolstr.2012.04.006>.
- [11]- L. Yao, P. He, S. Song, A perturbation stochastic finite-element method for groundwater flow models based on an undetermined-coefficients approach. *Hydrogeology Journal*, 18 (2010) 1603-9. <https://doi.org/10.1007/s10040-010-0639-4>.
- [12]- J. Paiboon, D.V. Griffiths, J. Huang, G.A. Fenton, Numerical analysis of effective elastic properties of geomaterials containing voids using 3D random fields and finite elements. *International Journal of Solids and Structures*, 50 (2013), 3233-41. <https://doi.org/10.1016/j.ijsolstr.2013.05.031>.
- [13]- J.D. Arregui-Mena, L. Margetts, P.M. Mummery, Practical Application of the Stochastic Finite Element Method. *Archives of Computational Methods in Engineering*, 23 (2016), 171-90. <https://doi.org/10.1007/s11831-014-9139-3>.
- [14]- S. Adhikari, C.S. Manohar. Dynamic analysis of framed structures with statistical uncertainties. *International Journal for Numerical Methods in Engineering*, 44(8) (1999), 1157-78. [https://doi.org/10.1002/\(SICI\)1097-0207\(19990320\)44:8%3C1157::AID-NME549%3E3.0.CO;2-5](https://doi.org/10.1002/(SICI)1097-0207(19990320)44:8%3C1157::AID-NME549%3E3.0.CO;2-5)
- [15]- I. Elishakoff, Y. Ren, *Finite element methods for structures with large stochastic variations*. OUP Oxford, 2003.
- [16]- A. Nouy. Recent Developments in Spectral Stochastic Methods for the Numerical Solution of Stochastic Partial Differential Equations. *Archives of Computational Methods in Engineering*, 16 (2009), 251-85. <https://doi.org/10.1007/s11831-009-9034-5>.
- [17]- G.I. Schuëller, H.J. Pradlwarter, Uncertain linear systems in dynamics: Retrospective and recent developments by stochastic approaches. *Engineering Structures*, 31 (2009) 2507-17. <https://doi.org/10.1016/j.engstruct.2009.07.005>.
- [18]- J.E. Hurtado, A.H. Barbat, Monte Carlo techniques in computational stochastic mechanics. *Archives of Computational Methods in Engineering*, 5 (1998) 3-29. <https://doi.org/10.1007/BF02736747>.
- [19]- N.V. Thuan, T.D. Hien. Variability in frequencies of vehicle vibration anlysis with muiltiple random variables. *Transport and Communications Science Journal*, 72 (2021) 215-26. <https://doi.org/10.47869/tcsj.72.2.7>.
- [20]- M. Shinozuka, G. Deodatis, Simulation of Stochastic Processes by Spectral Representation. *Applied Mechanics Reviews*, 44 (1991) 191-204. <https://doi.org/10.1115/1.3119501>.
- [21]- M. Shinozuka, G. Deodatis, Simulation of Multi-Dimensional Gaussian Stochastic Fields by Spectral Representation. *Applied Mechanics Reviews*, 49 (1996) 29-53. <https://doi.org/10.1115/1.3101883>.
- [22]- E. Vanmarcke, M. Shinozuka, S. Nakagiri, G.I. Schuëller, M. Grigoriu, Random fields and stochastic finite elements. *Structural Safety*, 3 (1986) 143-66. [https://doi.org/10.1016/0167-4730\(86\)90002-0](https://doi.org/10.1016/0167-4730(86)90002-0).
- [23]- S. Adhikari, A reduced spectral function approach for the stochastic finite element analysis. *Computer Methods in Applied Mechanics and Engineering*, 200 (2011) 1804-21. <https://doi.org/10.1016/j.cma.2011.01.015>.
- [24]- R. Casella. Monte Carlo statistical methods, University of Florida, 2008.

-
- [25]- E.A. Johnson, S.F. Wojtkiewicz, L.A. Bergman, B.F. Spencer, Observations with regard to massively parallel computation for Monte Carlo simulation of stochastic dynamical systems. *International Journal of Non-Linear Mechanics*, 32 (1997) 721-34. [https://doi.org/10.1016/S0020-7462\(96\)00097-2](https://doi.org/10.1016/S0020-7462(96)00097-2).
- [26]- G.I. Schuëller, Computational stochastic mechanics – recent advances. *Computers & Structures*, 79 (2001) 2225-34. [https://doi.org/10.1016/S0045-7949\(01\)00078-5](https://doi.org/10.1016/S0045-7949(01)00078-5).
- [27]- S.A. Ramu, R. Ganesan, Stability of stochastic Leipholz column with stochastic loading. *Archive of Applied Mechanics*, 62 (1992) 363-75. <https://doi.org/10.1007/BF00804597>
- [28]- R. Ganesan, V.K. Kowda, Buckling of Composite Beam-columns with Stochastic Properties. *Journal of Reinforced Plastics and Composites*, 24 (2005) 513-43. <https://doi.org/10.1177/0731684405045017>.
- [29]- D.T. Hang, N.X. Tung, D.N. Tien, Stochastic buckling analysis of non-uniform columns using stochastic finite elements with discretization random field by the point method. *Engineering, Technology Applied Science Research*, 12 (2022) 8458-62. <https://doi.org/10.48084/etasr>.
- [30]- T.D. Hien, A static analysis of nonuniform column by stochastic finite element method using weighted integration approach. *Transport and Communications Science Journal*, 71 (2020), 359-67. <https://doi.org/10.25073/tcsj.71.4.5>.
- [31]- J.V.Z. Kala, J. Jönsson, Random fields of initial out of straightness leading to column buckling. *Journal of Civil Engineering and Management*, 23 (2017), 902-13. <https://doi.org/10.3846/13923730.2017.1341957>.
- [32]- V.K. Kowda, Free-vibration and buckling of prismatic and thin-walled composite beam-columns with stochastic properties, Concordia University, 2002.
- [33]- H.X. Nguyen, T.D. Hien, J. Lee, H. Nguyen-Xuan, Stochastic buckling behaviour of laminated composite structures with uncertain material properties. *Aerospace Science and Technology*, 66 (2017), 274-83. <https://doi.org/10.1016/j.ast.2017.01.028>.
- [34]- P. Sasikumar, R. Suresh, S. Gupta, Stochastic finite element analysis of layered composite beams with spatially varying non-Gaussian inhomogeneities. *Acta Mechanica*, 225 (2014), 1503-22. <https://doi.org/10.1007/s00707-013-1009-9>.
- [35]- M.D. McKay, R.J. Beckman, W. Conover, A comparison of three methods for selecting values of input variables in the analysis of output from a computer code. *Technometrics*, 42 (2000), 55-61. <https://doi.org/10.1080/00401706.2000.10485979>.
- [36]- I.M. Sobol, Global sensitivity indices for nonlinear mathematical models and their Monte Carlo estimates. *Mathematics and Computers in Simulation*, 55 (2001), 271-80. [https://doi.org/10.1016/S0378-4754\(00\)00270-6](https://doi.org/10.1016/S0378-4754(00)00270-6).