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Free Vibration of Statically Deformed Bi-directionally Graded Timoshenko Beam on Elastic Foundation

Hareram Lohar ^a, Anirban Mitra ^{a*}, Sarmila Sahoo ^b

^a Mechanical Engineering Department, Jadavpur University, Kolkata 700032, Kolkata, India

^b Civil Engineering Department, Heritage Institute of Technology, Kolkata-700107, Kolkata, India

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ABSTRACT

In the present work, geometric nonlinearity is taken into consideration to study the free vibration response of nonlinearly deflected Timoshenko beam with bi-directional gradation, resting on an elastic foundation of Winkler type. Exponential variation of material properties is considered along the thickness and length direction. In order to carry out the present numerical study, the entire physical domain of analysis is converted into a normalized domain and suitable number of computational points are created on it to conduct a whole domain analysis. The adopted mathematical formulation is displacement based and semi-analytical. A unique methodology is used for the formulation, in which the static analysis is taken first under large amplitude static force, then a subsequent eigenvalue analysis is performed on the known statically deformed beam profile. Appropriate energy principles are utilized for both analyses to derive the equation of motion in matrix form, which are solved with the help of the algorithm of direct substitution method. Generated results are validated with the results of previously published paper. New sets of result are also presented as benchmark results to represent the impact of bi-directional gradation and elastic foundation on nonlinear vibration response.

1 Introduction

Functionally graded material (FGM) is a newly developed advanced material, in which two or more constituent materials are combined together in a certain volume fraction to obtain variable thermo-mechanical properties with changing coordinates. The smooth and continuous transition in composition of the constituent materials possess significant advantages over conventional laminated composites by reducing residual and thermal stresses, as well as, stress concentration. Reduction of highly localised stresses further reduces adhesive bond separation, matrix cracking and delamination, which are major shortcomings of laminated composites. The graded transition in materials also refines the distribution of material properties and provides high stiffness/strength to weight ratio. These unique characteristics together with the design and fabrication

* Corresponding author. Tel.: +91 9836380537.

E-mail address: anirban.mitra@jadavpuruniversity.in

ability of FGMs of fulfilling specific requirements have made these revolutionary materials useful in many engineering applications fields such as aerospace, civil, mechanical engineering, biomedical implants, defence, automotive, nuclear industries, optoelectronics [1-4] etc.

Due to vast variety of applications of these materials in engineering field, there is considerable interest among researchers to explore their thermo-mechanical/structural behaviour. However, existing articles in literature indicate that most of the research works involve FGMs with simple unidirectional material gradation [5-20]. But, the temperature distributions in components that are used in space shuttles, aircraft, nuclear reactors, etc., change in two or three directions [1]. Design of these components added new challenge to develop more advanced materials that can withstand high temperatures in multiple direction. Bi-directional graded (BFG) materials fall in these categories, where gradation is carried out in two mutually orthogonal directions[21]. Recent development in research works proved that BFG materials have a great potential to design, fabricate and optimize critical and complex problems under multi-functional requirements.

Nguyen and Lee [22, 23] developed a finite element model to study static, dynamic and buckling behaviour of thin-walled BFG beams. Rajasekaran and Khaniki [24] presented a comprehensive study to investigate dynamic behaviour of BFG beams with single/multi crack. Different material gradation models, such as power-law, sigmoidal and exponential, were used to model the beam. Li et al [25, 26] proposed different types of novel meshless methods to investigate the flexural behaviour of FG beams. Yang et al [27] introduced a novel nonlinear model of BFG nanobeams to study bending, buckling and vibrational behaviour. Karamanlı [28] employed third order shear deformation theory to investigate free vibrational behaviour of BFG beams for different set of boundary conditions. Trinh et al [29] investigated the free vibration behaviour of BFG microbeams based on the frame work of quasi-3D theory. Chen et al [30] studied the geometric nonlinear static and dynamic responses of BFG microbeams considering higher order shear deformation theory. The authors [31] also considered a BFG microbeam embedded in elastic medium to study size dependent dynamic stability effects. In a separate article, the authors [32] considered a porous BFG beam to study the porosity sensitivity effects in the nonlinear resonance behaviour. Truong et al [33] developed a novel approach to optimize the material of statically loaded BFG structures, in which the material gradation of the BFG structure was presented using 2D-NURBS functions. Tang and Ding [34] performed nonlinear analysis to study hygro-thermal dynamics of a BFG beam under combined action of longitudinal and transverse displacements. Tang et al [35] conducted a nonlinear free vibration analysis on BFG Euler–Bernoulli beams to observe the dependency of nonlinear frequencies on material property variation. In a separate study, the authors [36] presented a novel model of BFG Timoshenko nanobeams resting on the Pasternak foundation to determine dynamic interaction in between beam and foundation. Fariborz and Batra [37] utilized shear deformation theory to predict the natural frequencies of FGM circular beams. Yang et al [38] employed two-dimensional linear elastic theory to study the dynamic response of non-uniform BFG beams under a moving harmonic load. Flexural analysis of BFG nanobeam was carried out by Nazmul and Devnath [39] utilizing Eringen's nonlocal elasticity theory. Lal and Dangi [40] considered the influence of surface effect to study dynamic response of moderately thick BFG nanobeam. Ohab-Yazdi and Kadkhodayan [41] carried out a study to investigate the free vibration of porous BFG rotary nanobeam. Thang et al [42] employed nonlocal strain gradient theory to study the static and buckling behaviour of BFG Timoshenko nanobeams. Abo-bakr et al [43] utilized the principle of particle swarm optimization technique to optimize the lightweight design of BFG beams for maximum fundamental frequency and critical buckling load. Selmi and Mustafa [44] investigated the dynamic behaviour of BFG beams by using the continuous elements method.

In modern analysis and design of structures, elastically restrained or elastically supported beam acts as a crucial tool for modelling and analysis of various complex engineering problems such as railroad tracks, highway pavement and continuously supported pipelines etc. In this context, the domain of FG beam on elastic foundation is more advanced and recent. The importance of the problem is understood by large number of researchers who are working in this domain and an abundant number of research articles are available in literature. However, it is observed that most of the research works are concentrated on unidirectional FG beam resting on elastic foundation [45 - 65].

Due to excellent properties of bi-directional or multidirectional FG beam, the domain of bi-directional or multidirectional FG beam resting on elastic foundation is also equally important. But, only a few numbers of research works are available in literature in this domain. Rajasekaran and Khaniki [66] carried out forced vibration analysis on a non-uniform BFG microbeam embedded in variable elastic environment models under a moving harmonic load/mass. Modified couple stress theory was utilized for the mathematical formulation of the problem and the governing set of differential equations was derived by using Hamilton's principle. A similar type of work was carried out by Azandariani et al [67] on Euler–Bernoulli beam to study for nonlinear static behaviour of a BFG microbeam rested on a nonlinear elastic foundation.

The extensive literature survey reveals that the field of BFG beam is relatively newer. To design the components of space shuttles, aircrafts, nuclear reactors etc, where temperature distribution takes place in more than single directions, no other material will be as effective as BFG material. The unique thermo-mechanical properties of the BFG materials can be useful to make these components resilient in such environment. On the other hand, beam resting on elastic foundation act as an indispensable tool to effectively design various load-bearing components. In this context, BFG beams on elastic foundation is an advanced domain and very few research papers are found in existing literature. From application perspective of the domain is important and thorough investigation is needed. Hence, the present study is taken up to investigate the free vibration of statically deformed BFG beam resting on elastic foundation. The elastic foundation is modelled by considering parallel arrangement of linear springs of constant stiffness. Exponential gradation in both axial and transverse direction is considered for BFG beam. A whole domain analysis in a normalized coordinate frame is presented, in which energy principle is utilized to formulate the governing differential equations. For considering geometric nonlinearity (which arises due to large amplitude deformation) these governing differential equations come-out nonlinear equations. Hence, a numerical tool is utilized to solve these nonlinear equations. The presented results of the current study are compared with the results of previously published article for successful validation. New set of results are also presented to study the impact of BFG material gradation and elastic foundation on linear/nonlinear natural frequencies, backbone curves, linear/nonlinear mode shapes. To the best of the author's acknowledgement such kind of results are new and are not readily available in existing literature.

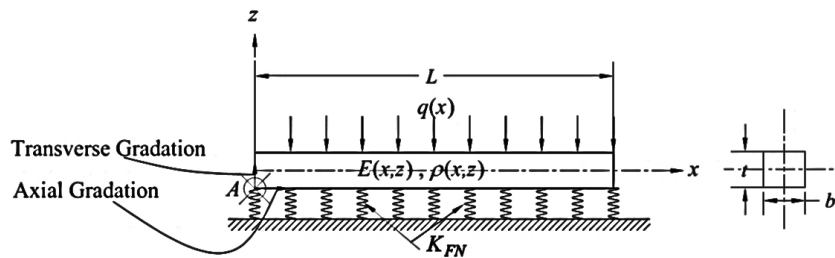


Fig. 1 – Representation of BFG beam resting on elastic foundation.

2 Mathematical Formulation

Fig. 1 shows a BFG beam of length L , width b and thickness t , resting on an elastic foundation of stiffness K_{FN} . A simplified Winkler foundation model is selected as elastic foundation, in which series of linear springs with constant stiffness are stacked on a rigid ground with parallel arrangement. The beam is carrying a uniformly distributed transverse load of magnitude $q(x)$ in vertically downward direction. The edge passing through the point 'A' is taken as the reference (root) edge for material gradation. From this edge onward, exponential variation of material properties (i.e., elastic modulus E , density ρ and Poisson's ratio μ) is obtained in both the axial and transverse direction of beam. Variation of the material properties in mathematical terms are expressed as follows [68].

$$E(x, z) = E_0 \exp\left(\frac{x}{L}\alpha_x + \frac{z}{t}\alpha_z\right)$$

$$\rho(x, z) = \rho_0 \exp\left(\frac{x}{L}\alpha_x + \frac{z}{t}\alpha_z\right)$$

$$\mu(x, z) = \mu_0 \exp\left(\frac{x}{L}\alpha_x + \frac{z}{t}\alpha_z\right)$$

In the above expression, E_0 , ρ_0 and μ_0 are root elastic modulus, density and Poisson's ratio, respectively, which are defined at the reference edge. α_x and α_z are gradation parameters for axial and transverse gradation, respectively.

The present formulation is displacement based semi-analytical technique, which employs variational form of energy principle to derive the governing equations of the system. At the starting point of the formulation, appropriate displacement fields are assumed depending on the nature of problem. These displacement fields are to be substituted in the expressions of energy functionals. Finally, energy principle is utilized to derive the governing set of differential equations. Determination of large amplitude free vibration response has two separate parts. At first part, the static problem is solved under large

amplitude external loading, which predicts the initial deformed shape of the beam. In the second part, a dynamic analysis is performed on the statically deflected system to represent the free vibration response.

The beam is modelled following Timoshenko beam theory, in which influences of both shear deformation and rotary inertia are included. This includes rotational (ψ) deformation in beam with bending (w) and stretching (u) deformations. These displacement fields are defined at the mid-plane of the beam. In large amplitude analysis, the expression of strains involves higher order term of displacement, which introduces nonlinearity into the formulation. The relations of axial strain (ε_{xx}) and shear strain (γ_{xz}) are mathematically expressed as [9, 69],

$$\varepsilon_{xx} = \frac{du}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 - z \frac{d\psi}{dx} \quad (1)$$

$$\gamma_{xz} = \frac{1}{2} \left(\frac{dw}{dx} - \psi \right) \quad (2)$$

The expression of Eq. (1) is clearly nonlinear in nature and the first two terms together represents von-Karman type of nonlinearity.

Substituting the strain-displacement expressions, the strain energy (U) can be expressed as follows,

$$U = \left[\frac{b}{2} \int_0^L \int_{-t/2}^{t/2} \left\{ \left(\frac{du}{dx} \right)^2 + \frac{1}{4} \left(\frac{dw}{dx} \right)^4 + z^2 \left(\frac{d\psi}{dx} \right)^2 + \frac{du}{dx} \left(\frac{dw}{dx} \right)^2 - z \left(\frac{dw}{dx} \right)^2 \frac{d\psi}{dx} - 2z \frac{du}{dx} \frac{d\psi}{dx} \right\} E(x, z) dx dz \right]_I + \left[\frac{bk_{sh}}{2} \int_0^L \int_{-t/2}^{t/2} \left\{ \left(\frac{dw}{dx} \right)^2 - 2 \frac{dw}{dx} \psi + \psi^2 \right\} G(x, z) dx dz \right]_{II} + \left[\frac{1}{2} \int_0^L K_{FN} w^2 dx \right]_{III} \quad (3)$$

In the above Eq. (3), the first two terms (I & II) are representing the strain energy of the beam due to axial and shear strain. On the other hand, the third term (III) of the expression is associated with the beam foundation. In the second term (II), k_{sh} is shear correction factor, which is taken as 5/6 for rectangular section [9] and $G(x, z)$ is shear modulus, related to elastic modulus by, $G(x, z) = E(x, z)/2(1+\mu)$.

Potential energy (V) due to external uniformly distributed load $q(x)$ can be represented as [69],

$$V = \int_0^L q(x) w dx \quad (4)$$

Kinetic energy (T) in the vibratory system can be written as [9, 69],

$$T = \frac{b}{2} \int_0^L \int_{-t/2}^{t/2} \left[\left(\frac{dw}{dt} \right)^2 + \left(\frac{du}{dt} \right)^2 + z^2 \left(\frac{d\psi}{dt} \right)^2 \right] \rho(x, z) dx dz \quad (5)$$

2.1 Static analysis

The static displacement fields (w , u and ψ ,) are assumed following Rayleigh-Ritz approximation [69],

$$w(x) = \sum_{i=1}^{nw} d_i \phi_i(x) \quad (6a)$$

$$u(x) = \sum_{i=1}^{nu} d_{nw+i} \alpha_i(x) \quad (6b)$$

$$\psi(x) = \sum_{i=1}^{nsi} d_{nw+nu+i} \beta_i(x) \quad (6c)$$

In the above expressions, d_i s are unknown parameters. ϕ_i , α_i and β_i are orthogonal admissible functions for w , u and ψ , respectively. These functions must be continuous and differentiable in the computational domain and fulfill the kinematic boundary conditions of the beam. nw , nu and nsi are the number for these functions, ϕ_i , α_i and β_i , respectively. The first of these functions (ϕ_1 , α_1 and β_1), which are known as start function, are furnished in Table 1 for various boundary conditions. These start functions are used in Gram-Schmidt orthogonalization algorithm to generate higher order functions.

Table 1 – List of start functions for the displacement fields [9]

Fields	Boundary	Conditions	Function
w	CC		
	CS	$w(0)=0, w(L)=0$	$\phi_1=(x/L)\{1-(x/L)\}$
	SS		
u	CC		
	CS	$u(0)=0, u(L)=0$	$\alpha_1=(x/L)\{1-(x/L)\}$
	SS		
ψ	CC	$\psi(0)=0, \psi(L)=0$	$\beta_1=\sin(\pi x/L)$
	CS	$\psi(0)=0, \psi(L) \neq 0$	$\beta_1=\sin(\pi x/2L)$
	SS	$\psi(0) \neq 0, \psi(L) \neq 0$	$\beta_1=\cos(\pi x/L)$

For static analysis, to derive the governing sets of equations minimum potential energy principle [69] is employed.

$$\delta(U+V)=0 \quad (7)$$

Substituting the expressions of the energy functionals, given by Eqs. (3), (4) and (5) into Eq. (7) and considering the assumed displacement fields, given by Eq. (6), the governing set of equations are obtained in matrix form, as follows,

$$[K]\{d\}=\{f\} \quad (8)$$

where, $[K]$ is the stiffness matrix having dimension $(nw + nu + nsi) \times (nw + nu + nsi)$. $\{d\}$ and $\{f\}$ are the vector of unknown parameters and load vector, respectively. These vectors also have a dimension of $(nw + nu + nsi)$. The form and elements of $[K]$ and $\{f\}$ are furnished in detail in the Appendix.

2.2 Dynamic analysis

A The assumed dynamic displacement fields are expressed as follows [69],

$$w(x, \tau) = \sum_{i=1}^{nw} c_i \phi_i(x) e^{j\omega\tau} \quad (9a)$$

$$u(x, \tau) = \sum_{i=1}^{nu} c_{nw+i} \alpha_i(x) e^{j\omega\tau} \quad (9b)$$

$$\psi(x, \tau) = \sum_{i=1}^{nsi} c_{nw+nu+i} \beta_i(x) e^{j\omega\tau} \quad (9c)$$

Dynamic displacement fields contain a spatial and a temporal part and it is assumed that these two parts are completely separable. Here, the spatial part is same as that considered in the static analysis and c_i s are a new set of unknown coefficients.

However, φ_i , α_i and β_i are same as the static problem. ω is frequency of the dynamic system and j indicates an imaginary number.

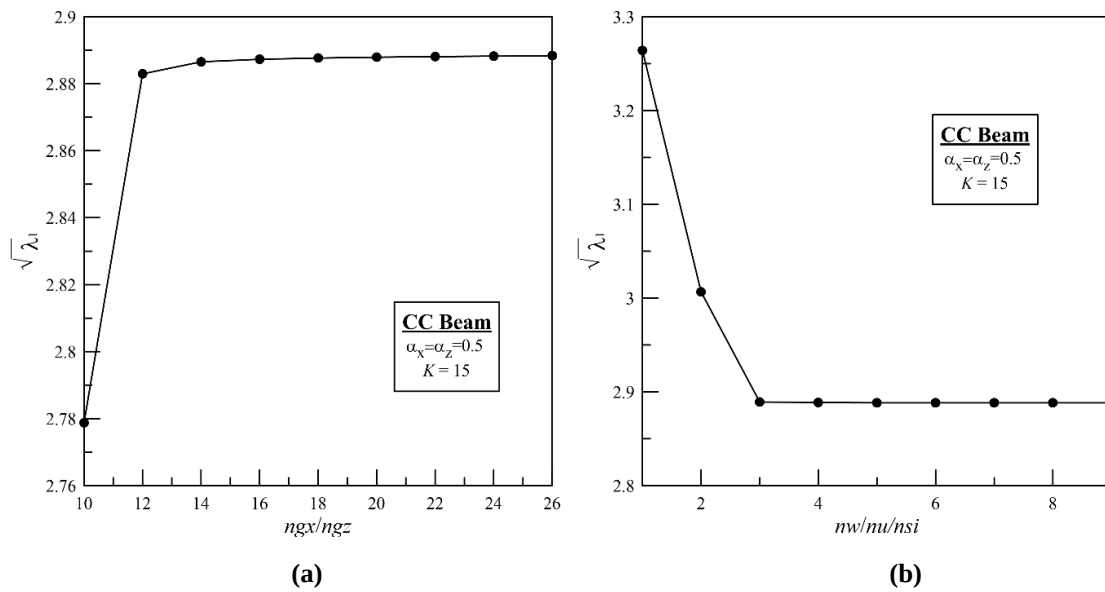


Fig. 2 – Convergence study for (a) gauss points (ngx/ngz) and (b) orthogonal functions ($nw/nu/nsi$)

For dynamic analysis, Hamilton's principle [69] is utilized to derive the governing sets of equations,

$$\delta \left(\int_{\tau_1}^{\tau_2} [T - (U + V)] d\tau \right) = 0 \quad (10)$$

Here, τ is the time. The expression of U remains unchanged from that described by Eq. (3) in static analysis. It is to be noted that present dynamic study is performed on deformed beam, whose static solution has already been obtained in the previous step. So, the potential energy of the external load (V) must be zero.

Substituting the expressions of energy functionals of Eq. (3) and Eq. (5), as well as the assumed dynamic displacements of Eq. (9) into Eq. (10), the governing equation is obtained as follows:

$$[K]\{c\} - \omega^2 [M]\{c\} = 0 \quad (11)$$

Here, $[K]$ is stiffness matrix of the system at deflected configuration and $[M]$ is mass matrix. The elements of $[M]$ are provided in Appendix.

3 Results and Discussions

In the present study, free vibrational behaviour of transversely loaded BFG beam resting on Winkler foundation, is performed. Geometric nonlinearity originating from large amplitude deformation is taken into consideration. The static analysis is carried out to obtain the deformed configuration, while a dynamic analysis is conducted on the statically deformed system to determine the loaded free vibrational responses. Backbone curve and mode shape plots are presented for different combinations of system parameters. The exponential gradation model is selected for the BFG beam, where variation of material properties is considered along both the mutually perpendicular length and thickness directions. To control the gradation, two different gradation parameters are considered in length and thickness directions, are α_x and α_z respectively. Elastic modulus, mass density and Poisson ratio at root are taken as 210 GPa, 7850 kg/m³ and 0.3, respectively, which resembles the property of structural steel. The length, breadth and thickness of the Timoshenko beam geometry are taken as 5m, 0.5m and 1m, respectively. To generate the results, following non-dimensional terms are used for foundation stiffness and natural frequency,

$$K = K_{FN} \left(\frac{L}{t_0} \right)^3 / E_0 b \quad \text{and} \quad \lambda = \left(\frac{\omega L^2}{t_0} \right) \sqrt{(\rho_0 / E_0)}$$

Table 2(a) – Comparison of non-dimensional fundamental natural frequencies (λ_1) of SS beam.

SS Beam	$L/h=5$		α_z				
	α_x	0.0	0.2	0.4	0.6	0.8	1.0
Simsek [68]	0.0	2.6767	2.6748	2.6669	2.6533	2.6337	2.6103
Present study		2.6727	2.6705	2.6693	2.6688	2.6687	2.6686
Simsek [68]	0.2	2.6767	2.6728	2.6650	2.6513	2.6318	2.6083
Present study		2.6695	2.6670	2.6654	2.6644	2.6638	2.6631
Simsek [68]	0.4	2.6728	2.6689	2.6611	2.6474	2.6279	2.6044
Present study		2.6653	2.6624	2.6603	2.6587	2.6574	2.6559
Simsek [68]	0.6	2.6650	2.6630	2.6552	2.6416	2.6220	2.5986
Present study		2.6599	2.6566	2.6540	2.6517	2.6496	2.6471
Simsek [68]	0.8	2.6572	2.6533	2.6455	2.6318	2.6123	2.5888
Present study		2.6535	2.6497	2.6464	2.6433	2.6402	2.6366
Simsek [68]	1.0	2.6455	2.6416	2.6337	2.6201	2.6005	2.5771
Present study		2.6458	2.6415	2.6374	2.6334	2.6292	2.6242

Table 2(b) – Comparison of non-dimensional fundamental natural frequencies (λ_1) of CC beam.

CC Beam	$L/h=5$		α_z				
	α_x	0.0	0.2	0.4	0.6	0.8	1.0
Simsek [68]	0.0	5.1943	5.1904	5.1806	5.1630	5.1396	5.1083
Present study		5.1738	5.1487	5.1126	5.0656	5.0075	4.9386
Simsek [68]	0.2	5.1962	5.1923	5.1826	5.1650	5.1396	4.1096
Present study		5.1539	5.1268	5.0887	5.0393	4.9788	4.9073
Simsek [68]	0.4	5.1982	5.1943	5.1845	5.1669	5.1435	5.1123
Present study		5.1333	5.1041	5.0635	5.0116	4.9484	4.8741
Simsek [68]	0.6	5.2041	5.2001	5.1904	5.1728	5.1474	5.1181
Present study		5.1118	5.0800	5.0386	4.9821	4.9159	4.8386
Simsek [68]	0.8	5.2119	5.2080	5.1962	5.1787	5.1552	5.1240
Present study		5.0886	5.0541	5.0080	4.9502	4.8808	4.8001
Simsek [68]	1.0	5.2197	5.2177	5.2060	5.1884	5.1650	5.1337
Present study		5.0634	5.026	4.9767	4.9153	4.8424	4.7581

Table 2(c) – Comparison of non-dimensional fundamental natural frequencies (λ_1) of CS beam.

CS Beam	$L/h=5$						
	α_x	0.0	0.2	0.4	0.6	0.8	1.0
Simsek [68]	0.0	3.8779	3.8759	3.8662	3.8505	3.8310	3.8037
Present study		3.8674	3.8546	3.8352	3.8090	3.7761	3.7370
Simsek [68]	0.2	3.8232	3.8212	3.8115	3.7958	3.7763	3.7490
Present study		3.8064	3.7932	3.7732	3.7466	3.7133	3.6735
Simsek [68]	0.4	3.7685	3.7666	3.7568	3.7431	3.7216	3.6962
Present study		3.7450	3.7312	3.7108	3.6836	3.6499	3.6095
Simsek [68]	0.6	3.7138	3.7099	3.7021	3.6865	3.6669	3.6416
Present study		3.6829	3.6686	3.6476	3.6201	3.5858	3.5450
Simsek [68]	0.8	3.6572	3.6552	3.6455	3.6318	3.6123	3.5869
Present study		3.6201	3.6053	3.5838	3.5556	3.5208	3.4796
Simsek [68]	1.0	3.6005	3.5986	3.5908	3.5751	3.5556	3.5322
Present study		3.5566	3.5412	3.5191	3.4903	3.4550	3.4132

Table 3(a) – Non-dimensional natural frequency of the first mode ($\sqrt{\lambda_1}$) of CC beam for different combination of gradation parameters (α_x and α_z) and foundation stiffness (K).

Foundation stiffness (K)	α_x	α_z					
		0.0	0.2	0.4	0.6	0.8	1.0
00	0.0	2.2764	2.2691	2.2611	2.2507	2.2377	2.2223
	0.2	2.2702	2.2643	2.2558	2.2448	2.2313	2.2152
	0.4	2.2657	2.2592	2.2502	2.2387	2.2245	2.2077
	0.6	2.2609	2.2539	2.2443	2.2321	2.2172	2.1997
	0.8	2.2558	2.2481	2.2379	2.2249	2.2093	2.1909
	1.0	2.2502	2.2419	2.2308	2.2171	2.2005	2.1813
10	0.0	2.3726	2.3588	2.3431	2.3256	2.3062	2.2848
	0.2	2.3600	2.3464	2.3309	2.3135	2.2941	2.2726
	0.4	2.3479	2.3345	2.3191	2.3016	2.2821	2.2604
	0.6	2.3363	2.3229	2.3074	2.2898	2.2701	2.2481
	0.8	2.3249	2.3114	2.2958	2.2780	2.2578	2.2354
	1.0	2.3136	2.3000	2.2841	2.2658	2.2452	2.2223
20	0.0	2.4598	2.4393	2.4173	2.3940	2.3691	2.3426
	0.2	2.4405	2.4207	2.3994	2.3766	2.3521	2.3259
	0.4	2.4223	2.4031	2.3823	2.3598	2.3356	2.3096
	0.6	2.4049	2.3862	2.3658	2.3534	2.3195	2.2935
	0.8	2.3883	2.3699	2.3496	2.3275	2.3034	2.2774
	1.0	2.3721	2.3539	2.3337	2.3116	2.2874	2.2610

Table 3(b) – Non-dimensional natural frequency of the first mode ($\sqrt{\lambda_1}$) of CS beam for different combination of gradation parameters (α_x and α_z) and foundation stiffness (K).

Foundation stiffness (K)	α_x	α_z					
		0.0	0.2	0.4	0.6	0.8	1.0
00	0.0	1.9666	1.9633	1.9584	1.9517	1.9432	1.9331
	0.2	1.9510	1.9476	1.9425	1.9356	1.9270	1.9166
	0.4	1.9352	1.9316	1.9263	1.9193	1.9105	1.8999
	0.6	1.9191	1.9154	1.9099	1.9026	1.8936	1.8828
	0.8	1.9027	1.8988	1.8931	1.8856	1.8764	1.8654
	1.0	1.8859	1.8818	1.8759	1.8682	1.8588	1.8475
10	0.0	2.1106	2.0953	2.0791	2.0619	2.0438	2.0247
	0.2	2.0840	2.0694	2.0539	2.0374	2.0198	2.0012
	0.4	2.0581	2.0441	2.0292	2.0132	1.9962	1.9780
	0.6	2.0326	2.0193	2.0049	1.9894	1.9728	1.9550
	0.8	2.0076	1.9948	1.9809	1.9659	1.9496	1.9321
	1.0	1.9830	1.9707	1.9572	1.9425	1.9265	1.9093
20	0.0	2.2301	2.2062	2.1818	2.1568	2.1314	2.1053
	0.2	2.1955	2.1729	2.1496	2.1258	2.1013	2.0762
	0.4	2.1621	2.1406	2.1184	2.0956	2.0720	2.0478
	0.6	2.1297	2.1092	2.0880	2.0661	2.0434	2.0199
	0.8	2.0982	2.0786	2.0583	2.0372	2.0153	1.9925
	1.0	2.0674	2.0488	2.0293	2.0089	1.9877	1.9655

The present analysis is a whole domain analysis and carried out in a normalized domain. To perform detailed computation, computational point (which are familiar as gauss points) are generated along the normalized length ($\xi = x/L$) and thickness directions ($\eta = z/t$). The selection of the number of gauss points (ngx/ngz) and number of orthogonal functions ($nw/nu/nsi$) are to be made carefully. The computational efficiency and accuracy greatly depend on the proper selection on these parameters. In this point, a suitable convergence study is important to perform. After performing the study, as shown in Fig. 2, these values are selected as 24 and 8 respectively.

Results generated through the adopted methodology of the present study is compared with the results of previously published articles of Simsek [68] for BFG Timoshenko beam. The comparison of non-dimensional fundamental frequency is tabulated in Table 2. The comparison shows that the generated results match satisfactorily with the established results.

Non-dimensional natural frequencies of the first mode (λ_1) for different combination of gradation parameters (α_x and α_z) and foundation stiffness value (K) are provided in Table 3. Three separate tables are furnished for three different boundary conditions. To generate the results, three different values of foundation stiffness are considered and those are 0, 10 and 20. The case of $K=0$ clearly indicates the condition of beam without foundation. Under each foundation stiffness, six different values of gradation parameters (α) are considered. The values are taken at a fixed interval of 0.2 while being varied from 0 to 1. It is important to note here that mathematical expression of material gradation is considered in such a way that zero value of gradation parameters (α_x and α_z) corresponds to homogeneous beam (without gradation) and non-zero values

correspond to bi-directionally graded beam. Additionally, the cases of [$\alpha_x = 0$ and $\alpha_z \neq 0$] and [$\alpha_x \neq 0$ and $\alpha_z = 0$] indicate the cases of transversely and axially graded beam, respectively.

From the Table 3, it is observed that for all possible combination natural frequency increases with the increase in stiffness of the foundation. This is quite an expected outcome as natural frequency of a system is proportional to the stiffness of the system. The rigidity of the beam edges also significantly affect the free vibrational response by making the system stiffer.

The highest value of natural frequency in CC beam and the lowest for SS beam justifies these statements. It is also observed that frequency has dependency on gradation parameters. Comparing the results, it is found that frequency decreases with increase of axial gradation parameter (α_x) when all other parameters are kept fixed. Similar type of results is found for transverse gradation parameter (α_z) as well. Here, presented mathematical expression for the material gradation dictates such type of trends in frequency.

With the increase of material gradation parameters, the material property such as mass, density and elastic modulus, increases. Further, the increment of these properties results in softening effects, which represents the decrement in frequency values. The effect of externally applied transverse pre-load on natural frequencies is the main concern of the present study. However, transverse load produces static deflection. Therefore, load dependency of the natural frequencies is presented in a normalised frequency (ω_{NL}/ω_L) vs. normalised maximum deflection (w_{max}/t_0) plane.

Table 3(c) – Non-dimensional natural frequency of the first mode ($\sqrt{\lambda_1}$) of SS beam for different combination of gradation parameters (α_x and α_z) and foundation stiffness (K).

Foundation stiffness (K)	α_x	α_z					
		0.0	0.2	0.4	0.6	0.8	1.0
00	0.0	1.6348	1.6342	1.6338	1.6337	1.6336	1.6335
	0.2	1.6339	1.6331	1.6326	1.6323	1.6321	1.6319
	0.4	1.6326	1.6317	1.6310	1.6306	1.6302	1.6297
	0.6	1.6309	1.6299	1.6291	1.6284	1.6278	1.6270
	0.8	1.6290	1.6278	1.6268	1.6258	1.6249	1.6238
	1.0	1.6266	1.6253	1.6240	1.6228	1.6215	1.6199
10	0.0	1.8621	1.8431	1.8250	1.8079	1.7918	1.7766
	0.2	1.8431	1.8252	1.8083	1.7923	1.7772	1.7630
	0.4	1.8249	1.8082	1.7923	1.7773	1.7632	1.7498
	0.6	1.8074	1.7918	1.7769	1.7629	1.7495	1.7369
	0.8	1.7907	1.7760	1.7621	1.7488	1.7362	1.7242
	1.0	1.7747	1.7609	1.7477	1.7352	1.7232	1.7118
20	0.0	2.0276	1.9983	1.9700	1.9427	1.9166	1.8915
	0.2	1.9984	1.9708	1.9440	1.9183	1.8936	1.8700
	0.4	1.9704	1.9443	1.9190	1.8947	1.8715	1.8492
	0.6	1.9435	1.9188	1.8949	1.8720	1.8501	1.8291
	0.8	1.9176	1.8943	1.8718	1.8502	1.8295	1.8097
	1.0	1.8927	1.8707	1.8495	1.8291	1.8095	1.7907

The resulting plots are known as backbone curves, which are shown in Figs. 3-5 for different combinations of system parameters. It is important to note that the fundamental frequencies (ω_1) at no load condition are the linear frequency (ω_L), which are used here to normalize nonlinear frequencies (ω_{NL}/ω_L). For all the cases, with the increasing load values the stiffness of the beam increases. Such type of occurrence describes the presence of geometric nonlinearity in the system. Moreover, increased stiffness increases vibration frequencies. So, the system exhibits hardening type of nonlinear behaviour for all combinations of system parameters.

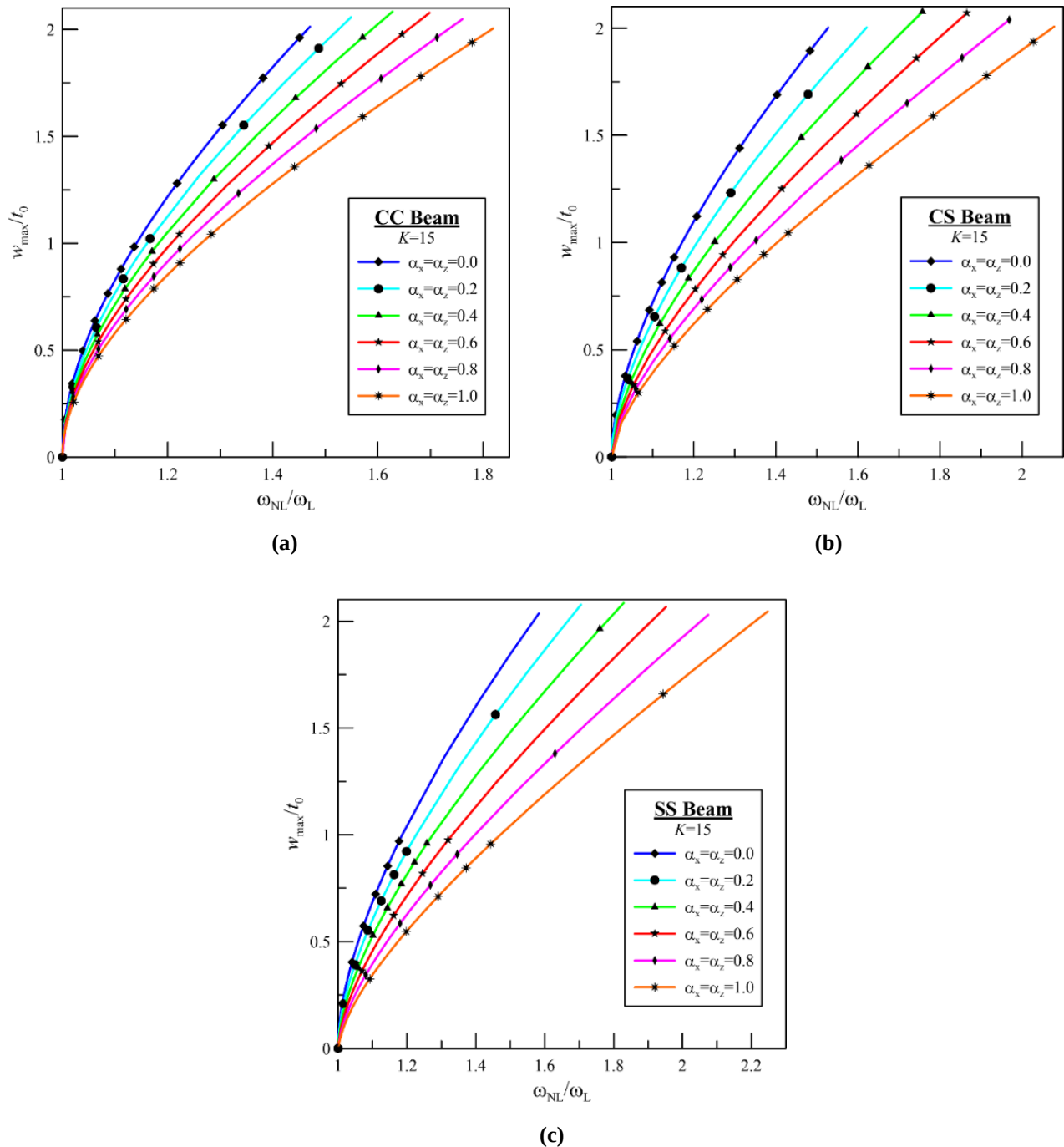


Fig. 3 – Backbone curves for different value of gradation parameters for (a) CC beam (b) CS beam and (c) SS beam.

Fig. 3 represents the backbone curves of the first mode for different combination of gradation parameters. Three sets of plots are shown for three different boundary condition and in each plot four backbone curves are presented for different combination of gradation parameters which varies from 0 to 1 with a fixed interval of 0.2, whereas foundation stiffness is

taken as 15. For effective gradation, the gradation parameters (α_x and α_z) must have some non-zero value. To compare the effect of gradation with non-graded (homogeneous) beam the result corresponds to $\alpha_x = \alpha_z = 0$ is presented. The values of both the gradation parameters could be different, but here same value ($\alpha_x = \alpha_z$) is assigned in order to keep the volume of results in check. From Fig. 3, it is observed that in all the cases with the increase in gradation parameter values the slope of the backbone curve is decreasing. The increasing value of gradation parameter gradually increases the mass of the beam system, which directly reduces the linear and nonlinear frequency of the system, which reflects in decrement of slope of backbone curve. This effect is more pronounced for SS boundary condition as SS beam is less stiff to deflect in loading condition, whereas, for CC beam backbone curves are closely clustered.

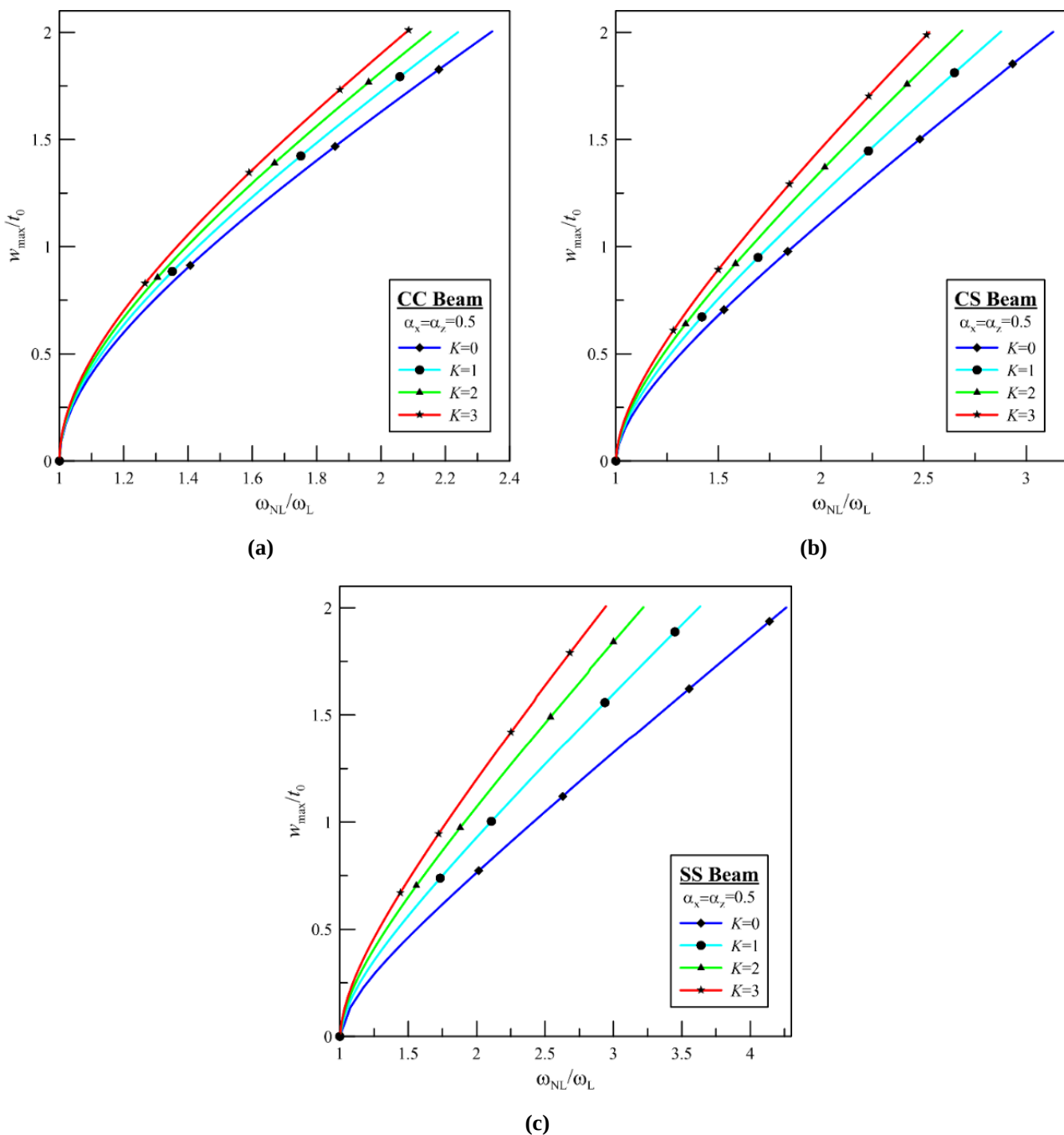


Fig. 4 – Backbone curves for different value of foundation stiffness for (a) CC beam (b) CS beam and (c) SS beam.

Backbone curves for different values of foundation stiffness are shown in Fig. 4. Here also three set of plots are presented for three different boundary conditions and in each plot four backbone curves are presented for different value of foundation

stiffness, which varies from 0 to 3, at a constant interval of 1. But both the gradation parameters are taken as 0.5. The foundation stiffness dependency of the backbone can be observed from the Fig. 4. It can be observed that for all cases with the increase of the foundation stiffness the slope of the backbone curve is increasing. This is due to the reason that higher value of foundation stiffness increases the frequency of the system and also makes the system stiffer to deflect under transverse loading.

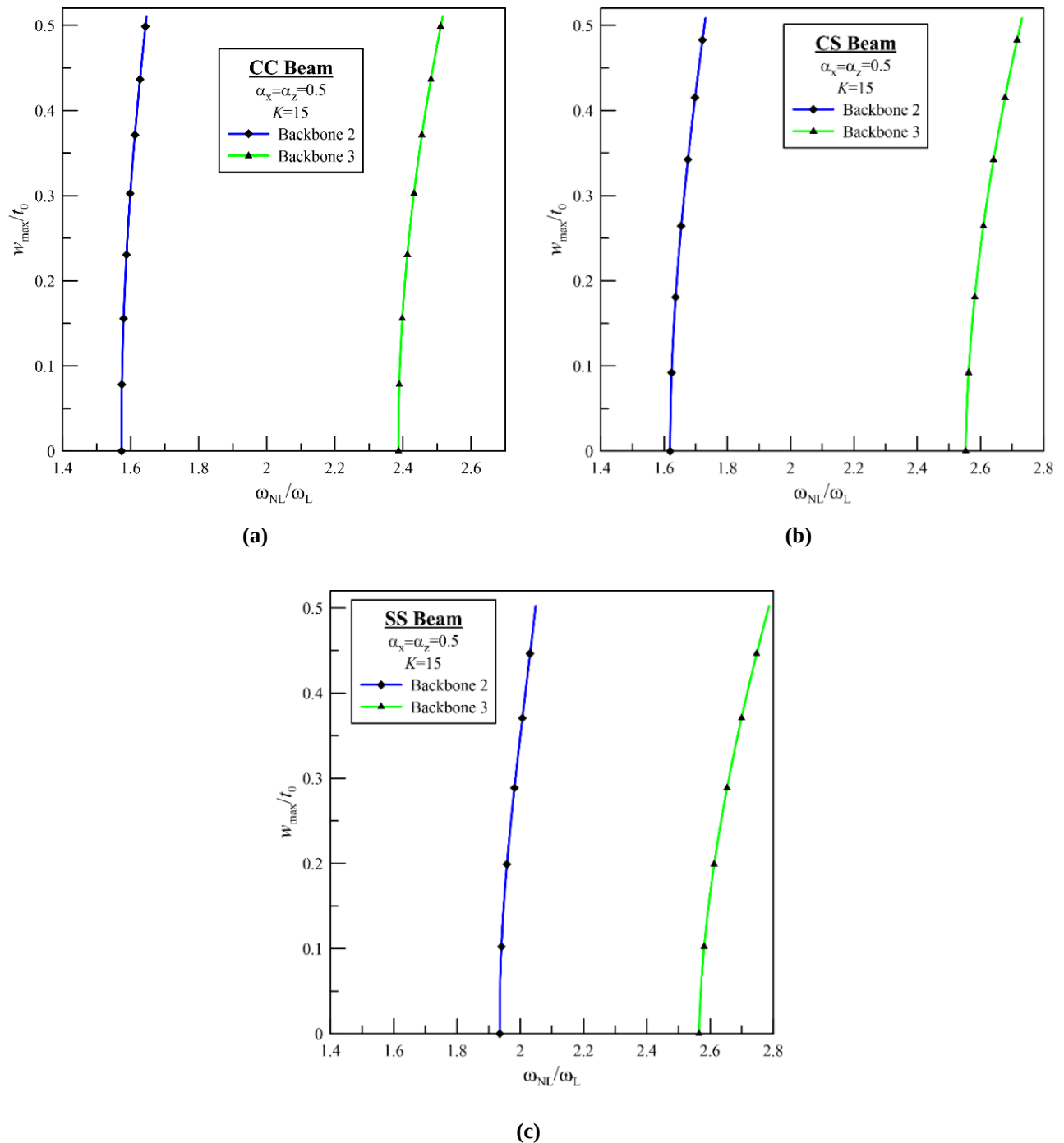


Fig. 5 – Higher order backbone curves (Mode 2 & 3) for (a) CC beam (b) CS beam and (c) SS beam.

Higher mode backbone curves are also important for nonlinear analysis of continuous system. The amount of nonlinearity involved at higher mode can be predicted from these plots. Backbone curves for modes 2 & 3 corresponding to three different boundary conditions are shown in Fig. 5. To generate the results, foundation stiffness and both the gradation

parameters are given the following values - 15 and 0.5, respectively. It is to be noted that backbone curves beyond mode 4 are also possible. But those curves are not shown here to maintain the clarity of the plots.

First three mode shapes corresponding to three different boundary conditions, are plotted in a normalized plane as shown in Fig. 6. For each vibration modes two distinct mode shapes are shown. One is linear mode shape, obtained at no load condition ($w_{\max}/t_0 = 0$) and other one is nonlinear mode shape, obtained at loaded condition ($w_{\max}/t_0 = 0.5$). The difference between these two measures the amount of nonlinearity presents in the system. To generate the results, foundation stiffness and both the parameters are kept as constant, which are 15 and 0.5, respectively.

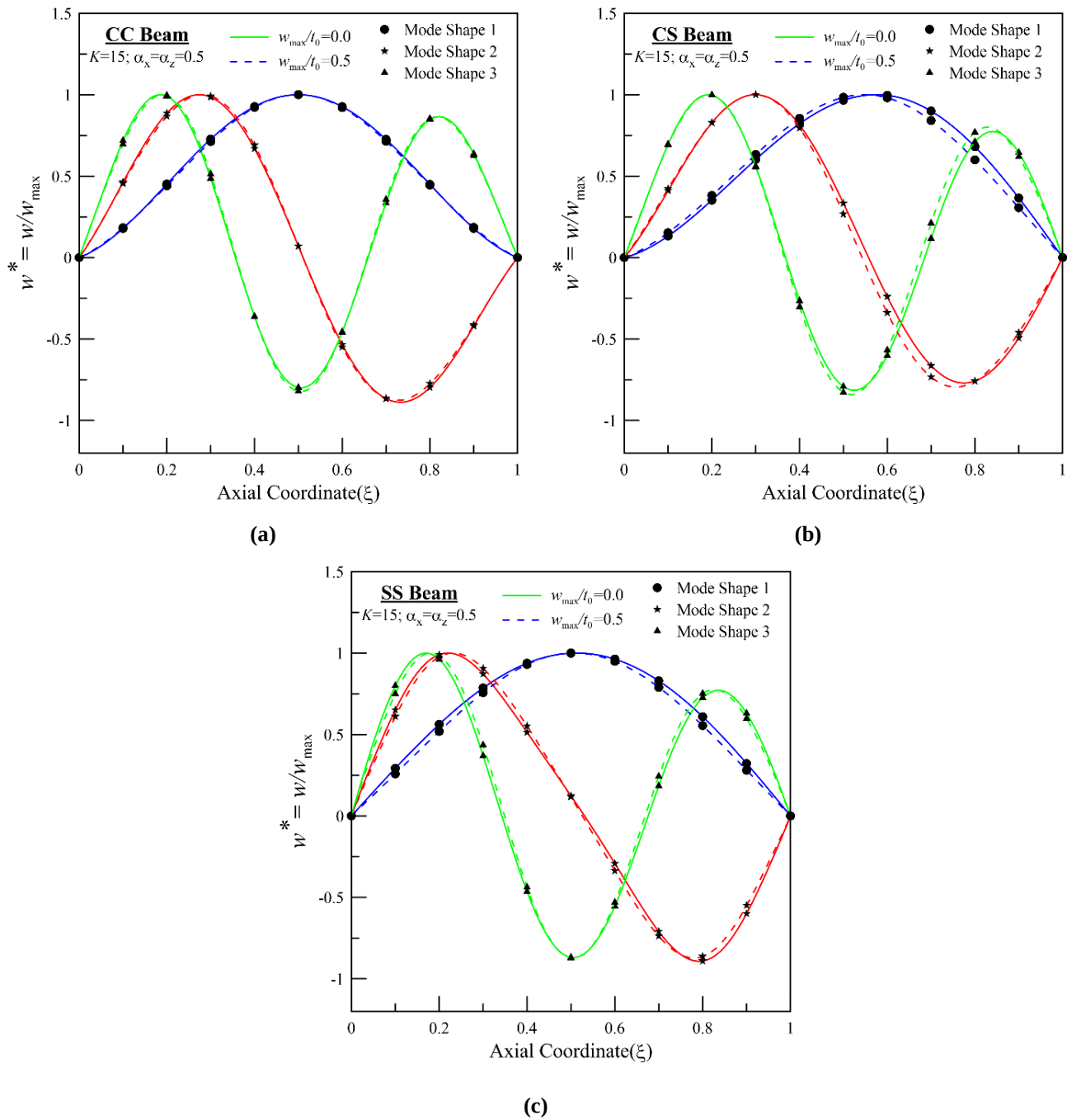


Fig. 6 – First three linear and nonlinear mode shape for (a) CC beam (b) CS beam and (c) SS beam.

4 Conclusions

The present study is conducted on a BFG Timoshenko beam which rests on an elastic foundation to investigate the nonlinear free vibration of the system. Exponential variation of the material properties is considered in both the transverse and axial direction. Nonlinearity associated with the large amplitude deformation under the application of uniformly distributed load is presented. A unique way is adopted for the formulation, in which a small amplitude free vibration analysis is conducted on largely statically deformed beam configuration. Thus, finally a large amplitude dependent free vibration response is derived. Variational form of suitable energy principle is utilized to derive the governing differential equations. The derived differential equations appear as function of unknown coefficients, whose solution is obtained by the implication of a numerical technique of direct substitution with relaxation. The presented entire analysis is a whole domain analysis and performed in normalized domain. The assumed methodology of the present paper is validated by comparing the results of the present study with the generated results of previously published research article. New sets of result are also presented to study the effect of material gradation and elastic foundation on free vibrational behaviour of the system. For different set of parameters loaded/no-loaded natural frequencies, backbone curves and mode shape plots are obtained for different boundary conditions.

The important observations of the present study can be summed up as,

- (i) The high value of foundation stiffness of elastic foundation effectively increases the natural frequency of the system. But the high value of gradation parameters in the adopted material model decreases the natural frequency of the system.
- (ii) The slope of the backbone curve decreases with the increasing value of material gradation parameters and increases with the increasing value of foundation stiffness.
- (iii) In all possible combination, nonlinearity present in SS beam is maximum whereas in CC beam is minimum.
- (iv) In each and every case of backbone curve, hardening type of geometric nonlinear characteristics is observed.

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Appendix A.

The elements of the stiffness matrix $[K_{ij}]$ are:

$$\begin{aligned}
 [K_{ij}]_{j=1,nw}^{i=1,nw} &= \frac{bh}{2L^3} \int_0^1 \int_0^1 \left(\sum_{k=1}^{nw} d_k \frac{d\phi_k}{d\xi} \right)^2 \frac{d\phi_i}{d\xi} \frac{d\phi_j}{d\xi} E(\xi, \eta) d\xi d\eta + \frac{bh}{L^2} \int_0^1 \int_0^1 \left(\sum_{k=nw+1}^{nw+nu} d_k \frac{d\phi_{k-nw}}{d\xi} \right) \frac{d\phi_i}{d\xi} \frac{d\phi_j}{d\xi} E(\xi, \eta) d\xi d\eta \\
 &\quad - \frac{bh^2}{L^2} \int_0^1 \int_0^1 \left(\sum_{k=nw+nu+1}^{nw+nu+nsi} d_k \frac{d\beta_{k-nw-nu}}{d\xi} \right) \frac{d\phi_i}{d\xi} \frac{d\phi_j}{d\xi} \left(\eta - \frac{1}{2} \right) E(\xi, \eta) d\xi d\eta + \frac{bhk_{sh}}{L} \int_0^1 \int_0^1 \frac{d\phi_i}{d\xi} \frac{d\phi_j}{d\xi} G(\xi, \eta) d\xi d\eta + KL \int_0^1 q(\xi) \phi_i \phi_j d\xi \\
 [K_{ij}]_{j=nw+1, nw+nu}^{i=1, nw} &= \frac{bh}{2L^2} \int_0^1 \int_0^1 \left(\sum_{k=1}^{nw} d_k \frac{d\phi_k}{d\xi} \right) \frac{d\phi_i}{d\xi} \frac{d\alpha_{j-nw}}{d\xi} E(\xi, \eta) d\xi d\eta \\
 [K_{ij}]_{j=nw+nu+1, nw+nu+nsi}^{i=1, nw} &= -bhk_{sh} \int_0^1 \int_0^1 \frac{d\phi_i}{d\xi} \beta_{j-nw-nu} G(\xi, \eta) d\xi d\eta - \frac{bh^2}{2L^2} \int_0^1 \int_0^1 \left(\sum_{k=1}^{nw} d_k \frac{d\phi_k}{d\xi} \right) \frac{d\phi_i}{d\xi} \frac{d\beta_{j-nw-nu}}{d\xi} \left(\eta - \frac{1}{2} \right) E(\xi, \eta) d\xi d\eta \\
 [K_{ij}]_{j=1, nw}^{i=nw+1, nw+nu} &= 0 \\
 [K_{ij}]_{j=nw+1, nw+nu}^{i=nw+1, nw+nu} &= \frac{bh}{L} \int_0^1 \int_0^1 \frac{d\alpha_{i-nw}}{d\xi} \frac{d\alpha_{j-nw}}{d\xi} E(\xi, \eta) d\xi d\eta \\
 [K_{ij}]_{j=nw+nu+1, nw+nu+nsi}^{i=nw+1, nw+nu} &= -\frac{bh^2}{L} \int_0^1 \int_0^1 \frac{d\alpha_{i-nw}}{d\xi} \frac{d\beta_{j-nw-nu}}{d\xi} \left(\eta - \frac{1}{2} \right) E(\xi, \eta) d\xi d\eta \\
 [K_{ij}]_{j=1, nw}^{i=nw+nu+1, nw+nu+nsi} &= -bhk_{sh} \int_0^1 \int_0^1 \beta_{i-nw-nu} \frac{d\phi_j}{d\xi} G(\xi, \eta) d\xi d\eta \\
 [K_{ij}]_{j=nw+nu+1, nw+nu}^{i=nw+nu+1, nw+nu+nsi} &= -\frac{bh^2}{L} \int_0^1 \int_0^1 \frac{d\beta_{i-nw-nu}}{d\xi} \frac{d\alpha_{j-nw}}{d\xi} \left(\eta - \frac{1}{2} \right) E(\xi, \eta) d\xi d\eta \\
 [K_{ij}]_{j=nw+nu+nsi}^{i=nw+nu+1, nw+nu+nsi} &= \frac{bh^3}{L} \int_0^1 \int_0^1 \frac{d\beta_{i-nw-nu}}{d\xi} \frac{d\beta_{j-nw-nu}}{d\xi} \left(\eta - \frac{1}{2} \right)^2 E(\xi, \eta) d\xi d\eta \\
 &\quad + bhLk_{sh} \int_0^1 \int_0^1 \beta_{i-nw-nu} \beta_{j-nw-nu} G(\xi, \eta) d\xi d\eta
 \end{aligned}$$

The elements of the load vector $\{f_i\}$ are,

$$\{f_i\}_{i=1, nw} = qL \int_0^1 \phi_i d\xi$$

$$\{f_i\}_{i=nw+1, nw+nu} = 0$$

$$\{f_i\}_{i=nw+nu+1, nw+nu+nsi} = 0$$

The elements of the mass matrix are $[M_{ij}]$:

$$[M_{ij}]_{\substack{i=1,nw \\ j=1,nw}} = Lbh \int_0^1 \int_0^1 \phi_i \phi_j \rho(\eta, \xi) d\eta d\xi$$

$$[M_{ij}]_{\substack{i=nw+1,nw+nu \\ j=nw+1,nw+nu}} = Lbh \int_0^1 \int_0^1 \alpha_{i-nw} \alpha_{j-nw} \rho(\eta, \xi) d\eta d\xi$$

$$[M_{ij}]_{\substack{i=nw+nu+1,nw+nu+nsi \\ j=nw+nu+1,nw+nu+nsi}} = Lbh^3 \int_0^1 \int_0^1 \beta_{i-nw-nu} \beta_{j-nw-nu} \rho(\eta, \xi) d\eta d\xi$$

Rest all other elements of the mass matrix $[M_{ij}]$ are zero.