

BAYESIAN ESTIMATION OF THE PARAMETERS OF THE WEIBULL LOG-LOGISTIC MIXTURE DISTRIBUTIONS USING TIME CENSORED DATA

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ABSTRACT: This paper is about studying the mixture of the Weibull-log-logistic distributions under Bayesian perspective. This model can be considered as new class of flexible models for heterogeneous lifetime data. Parameters of the model are estimated using the Gibbs sampling technique under type I censoring scheme. The behaviour of the Bayes estimators is studied assuming different priors. Finally, the model is applied to a real data set.

Keywords: Weibull distribution, Log-logistic distribution, finite mixture distributions, Gibbs sampling technique, Type I censoring

JEL Classification: C13 C16

1. INTRODUCTION:

A finite mixture distribution is a weighted sum of component distributions which has been frequently used to model a wide variety of random phenomena and specially to account for unobserved heterogeneity in data. Theory and applications of mixture models can be found in [1, 2]. In life testing it often happens that there is more than one cause of failure which may create heterogeneous populations. There can be different causes of failure and each failure mode has its own probability distribution [3]. Suppose that one cause has the Weibull distribution ($W(\alpha_1, \beta_1)$) and the other one has the Log-Logistic distribution ($LLG(\alpha_2, \beta_2)$), mixing these two distributions using a mixing proportion result the Weibull-Log-logistic mixture distributions which is useful to model multiple causes of failure simultaneously.

The mixture of Weibull-Log logistic distributions (MWLLG) has its pdf as:

$$f(t, \psi) = pf_1(t, \psi_1) + (1 - p)f_2(t, \psi_2) \quad 0 < p < 1 \quad (1)$$

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where:

$$f_1(t, \psi_1) = \left(\frac{\beta_1}{\alpha_1}\right) \left(\frac{t}{\alpha_1}\right)^{\beta_1-1} e^{-\left(\frac{t}{\alpha_1}\right)^{\beta_1}}, t > 0, \alpha_1 > 0, \beta_1 > 0 \quad (2)$$

is the Weibull density function with scale parameter $\alpha_1 = \lambda^{-1}$, and shape parameter β_1 , and:

$$f_2(t, \psi_2) = \frac{\left(\frac{\beta_2}{\alpha_2}\right) \left(\frac{t}{\alpha_2}\right)^{\beta_2-1}}{\left[1 + \left(\frac{t}{\alpha_2}\right)^{\beta_2}\right]^2}, \alpha_2 > 0, \beta_2 > 0, t > 0 \quad (3)$$

is the Log -Logistic density function, with shape α_2 and scale β_2 . where:

$\psi = (p_1, \alpha_1, \alpha_2, \beta_1, \beta_2)$, $\psi_i = (\alpha_i, \beta_i)$, $i = 1, 2$ are parameters vectors, and $\sum_{i=1}^2 p_i = 1$. Evidently, the reliability function of the mixture is given by:

$$R(t, \psi) = pR_1(t, \psi_1) + (1 - p)R_2(t, \psi_2) \quad (4)$$

where:

$$R_1(t, \psi_1) = \exp\left(-\left(\frac{t}{\alpha_1}\right)^{\beta_1}\right), t > 0, \alpha_1 > 0, \beta_1 > 0 \quad (5)$$

$$R_2(t, \psi_2) = \frac{1}{1 + (t/\alpha_2)^{\beta_2}}, \alpha_2 > 0, \beta_2 > 0, t > 0. \quad (6)$$

2. GIBBS SAMPLER

This Procedure aims to construct a Markov Chain whose values converge to the target posterior distribution by considering simple univariate conditional distributions. This Procedure is appropriate in situations where the joint distribution is not known explicitly while the full conditional distributions corresponding to each random variable are known and tractable.

Consider a bivariate random variable (x, y) and suppose that the aim is to compute one or both marginals, $p(x)$ and $p(y)$; so it is easier to consider a sequence of conditional distributions, $p(x|y)$ and $p(y|x)$, than obtaining the marginal by integration of the joint density $p(x, y)$. The sampler generate a random variable x_0 from the conditional distribution $p(x|y = y_0)$ using some initial value y_0 . The value of the generated variable x_0 will be used after that to generate a new value of y_1 . This process can be expressed as follow :

$$\begin{aligned} y_i &= p(y|x = x_0) \\ x_i &= (x|y = y_i - 1) \end{aligned}$$

By repeating the process "k" times, the Gibbs sequence converges after some burn-in to a stationary (equilibrium) distribution that is independent of the starting values (the target distribution). In order to implement the Gibbs Sampler, the following steps are required:

1. Derive full conditional distribution of the unknown parameters from the posterior.
2. The sampler starts with some initial values. It is better to select values that are close to the centre of the posterior like MLE's.
3. Draw from the full conditional distributions as explained before.
4. Estimate the parameters of interest by applying a Monte Carlo estimation approach to the generated samples.

In our research the model was implemented using the "OpenBUGS" software package [4]. The OpenBUGS software was run from R using the package "R2OpenBUGS" [5] for enhanced outputs. Note that the BUGS languages use different parametrizations of common probability density functions; where the Weibull distribution has a shape, rate parametrizations in BUGS language contrary to our case where we use shape, scale parametrization. Thus a translation technique has been used to obtain Bayes estimators that are convenient to our case. [6]. We also specify new prior distributions especially for " α_1 " [7]. Once the model was implemented and compiled the software will compute all the required univariate marginal's.

3. PRIOR DISTRIBUTION AND POSTERIOR ANALYSIS

In this paper we considered two type of data; Postmortem data : in which the times to failure of the item under study and it's failure causes are known ,and non-postmortem data : in which the failure causes is not known and the observations are unclassified [8].

We assume that the scale parameters α_i has a gamma prior distribution with the shape and scale parameters as c and d, respectively with pdf :

$$\pi(\alpha_i | c_i, d_i) \propto \alpha_i^{c_i-1} \exp(-d_i \alpha_i)$$

and we assume that the shape β_i has a gamma prior distribution with the shape and scale parameters as k and m, respectively with pdf :

$$\pi(\beta_i | e_i, m_i) \propto \beta_i^{k_i-1} \exp(-m_i \beta_i)$$

and we assume also that the mixing proportion p has a beta prior distribution with the shape and scale parameters as s_1 and s_2 , respectively with pdf :

$$\pi(p) \propto p^{s_1-1} (1-p)^{s_2-1}$$

Thus the joint prior of ψ : the vector of all unknown parameters is given by :

$$\pi(\psi) \propto \alpha_1^{c_1-1} \alpha_2^{c_2-1} \beta_1^{k_1-1} \beta_2^{k_2-1} (1-p)^{s_2-1} p^{s_1-1} e^{-\alpha_1 d_1 - \alpha_2 d_2 - \beta_1 m_1 - \beta_2 m_2} \quad (7)$$

3.1. Bayesian estimation under type I censored postmortem data

For a test sample size of n items and for a number of sub-populations ' $C = 2$ '. Suppose that the test is stooped at a predetermined time " T^* ". Let r_1 represent the number of failed items for the first subpopulation, and r_2 for the second one, where for a total of failed items $r = r_1 + r_2$, we have $(n - r)$ survived items. In view of result of [9] the likelihood function is given by:

$$L = \frac{n!}{(n-r)!} P^{r_1} (1-p)^{r_2} (\prod_{i=1}^{r_1} f_1) (\prod_{i=1}^{r_2} f_2) (R(T^*))^{n-r} \quad (8)$$

Replacing f_1 , f_2 and $R(T^*)$ by theirs formulas given in (2), (3) and (4) respectively in (8) we get :

$$L = \frac{n!}{(n-r)!} P^{r_1} (1-p)^{r_2} \left(\frac{\beta_1}{\alpha_1}\right)^{r_1} \prod_{i=1}^{r_1} \left(\frac{t_i}{\alpha_1}\right)^{\beta_1-1} \exp\left(-\sum_{i=1}^{r_1} \left(\frac{t_i}{\alpha_1}\right)^{\beta_1}\right) \left(\frac{\beta_2}{\alpha_2}\right)^{r_2} \prod_{i=1}^{r_2} \left(\frac{t_i}{\alpha_2}\right)^{\beta_2-1} \prod_{i=1}^{r_2} \left(1 + \left(\frac{t_i}{\alpha_2}\right)^{\beta_2}\right)^{-2} \left(p \exp\left[-\left(\frac{T^*}{\alpha_1}\right)^{\beta_1}\right] + (1-p) \left(\frac{1}{1 + (T^*/\alpha_2)^{\beta_2}}\right)\right)^{(n-r)} \quad (9)$$

Combining the likelihood under type I censored post-mortem data given by formula (9) with the prior given in (7) using Bayes' theorem, the joint posterior of ψ is given by:

$$\pi(\psi|t) \propto f(t|\psi) \pi(\psi)$$

$$\pi(\psi|t) \propto \frac{n!}{(n-r)!} p^{r_1+s_1-1} (1-p)^{r_2+s_2-1} \alpha_1^{c_1-r_1-1} \alpha_2^{c_2-r_2-1} \beta_1^{k_1+r_1-1} \beta_2^{k_2+r_2-1} \exp\left(\beta_1 \sum_{i=1}^{r_1} \ln\left(\frac{t_i}{\alpha_1}\right) + \beta_2 \sum_{i=1}^{r_2} \ln\left(\frac{t_i}{\alpha_2}\right)\right) \exp\left(-\sum_{i=1}^{r_1} \ln\left(\frac{t_i}{\alpha_1}\right) - \sum_{i=1}^{r_2} \ln\left(\frac{t_i}{\alpha_2}\right)\right) \exp\left(-\sum_{i=1}^{r_1} \left(\frac{t_i}{\alpha_1}\right)^{\beta_1}\right) \exp\left(-2 \sum_{i=1}^{r_2} \ln\left(1 + \left(\frac{t_i}{\alpha_2}\right)^{\beta_2}\right)\right) \exp\left(-\alpha_1 d_1 - \alpha_2 d_2 - \beta_1 m_1 - \beta_2 m_2 + (n-r) \ln(R(T^*))\right) \quad (10)$$

In addition Under the Squared Error Loss (SEL) function ; $L_s(\hat{\phi}, \phi) = (\hat{\phi} - \phi)^2$, The Bayes estimator of any function of α and β say $g(\alpha, \beta)$ is the posterior expectation,

$$\hat{g}_s = \frac{\int_0^\infty \int_0^\infty g(\alpha, \beta) \pi(\psi|t) d\alpha d\beta}{\int_0^\infty \int_0^\infty \pi(\psi|t) d\alpha d\beta} \quad (11)$$

Unfortunately, There is no closed form of the ratio of the two integrals in (11). Therefore we propose using the Gibbs sampling procedure to approximate (11) and also construct the corresponding credible intervals. The full conditional posterior PDFs for each unknown parameter are required for the implementation of the Gibbs sampling. Using (10) we derive the conditional posterior PDFs for each unknown parameter as follow:

3.1.1. The conditional distribution of the mixing proportion(p)

We do not write the term which does not involve p in (10) . The conditional distribution of the mixing proportion becomes:

$$\pi(p|\beta_1, \beta_2, \alpha_1, \alpha_2, t) \propto p^{r_1+s_1-1} (1-p)^{r_2+s_2-1} (R(T^*))^{(n-r)} \quad (12)$$

A fundamental technique that can be used to generate i.i.d from any target probability distribution is the adaptive rejection sampling method(ARS)[10]. However, this method can only be used with log-concave target densities, and it is preferred because of its high acceptance rates [11]. It is clear that the conditional density of p given in (12) doesn't have any standard form which require the study of its concavity. We study the concavity of the conditional distribution (12). Thus its logarithm is as follow:

$$\ln (\pi(p|\beta_1, \beta_2, \alpha_1, \alpha_2, t)) = (r_1 + s_1 - 1) \ln(p) + (r_2 + s_2 - 1) \ln(1-p) + (n-r) \ln ((R(T^*))) \quad (13)$$

Thus the second derivative with respect to p is :

$$\frac{\partial^2 \ln (\pi(p|\beta_1, \beta_2, \alpha_1, \alpha_2, t))}{\partial p^2} = - \left(\frac{r_1 + s_1 - 1}{p^2} + \frac{r_2 + s_2 - 1}{1-p^2} + (n-r) \left(\frac{R_1(T^*) + R_2(T^*)}{R(T^*)} \right)^2 \right)$$

which is strictly negative, thus p can be generated using the (ARS) method

3.1.2. The conditional distribution of the scale parameter (α_1)

We do not write the term which does not involve α_1 in (10) .

The conditional distribution of the scale parameter (α_1) becomes:

$$\pi(\alpha_1|\beta_1, \beta_2, \alpha_2, p, t) \propto \alpha_1^{c_1-r_1-1} \exp ((\beta_1 - 1) \sum_{i=1}^{r_1} \ln (\frac{t_i}{\alpha_1}) - \sum_{i=1}^{r_1} (\frac{t_i}{\alpha_1})^{\beta_1} - \alpha_1 d_1 + (n-r) \ln (R(T^*))) \quad (14)$$

Since the conditional distribution of the scale parameter (α_1) given in (14), doesn't has any standard form, and in view of result of ([12]) we suggest a Gamma distribution $G(c_1 - r_1, \lambda)$ as a proposal(source) distribution. The parameter λ is chosen such that it gives the maximum acceptance rate ($AR(\alpha_1, \lambda)$)

$$AR(\alpha_1, \lambda) = \frac{\pi(\alpha_1|\beta_1, \beta_2, \alpha_2, p, t)}{G(c_1 - r_1, \lambda)} = \frac{\pi(\alpha_1|\beta_1, \beta_2, \alpha_2, p, t)}{\frac{e^{-\frac{\alpha_1}{\lambda}} \alpha_1^{c_1-r_1-1}}{\lambda^{c_1-r_1} \Gamma(c_1 - r_1)}} \quad (15)$$

Thus the logarithm of (15) is given by:

$$\ln(AR(\alpha_1, \lambda)) = \frac{\alpha_1}{\lambda} - \alpha_1 d_1 + (\beta_1 - 1) \sum_{i=1}^{r_1} \ln\left(\frac{t_i}{\alpha_1}\right) - \sum_{i=n}^{r_1} \left(\frac{t_i}{\alpha_1}\right)^{\beta_1} + (c_1 - r_1) \log(\lambda) + \log[\Gamma(c_1 - r_1)] + (n - r) \log(R(T^*)) \quad (16)$$

differentiating (16) with respect to α_1 and λ and equating the results to 0 we get :

$$\sum_{i=1}^{r_1} \frac{\beta_1 t_i \left(\frac{t_i}{\alpha_1}\right)^{\beta_1-1}}{\alpha_1^2} + \frac{1}{\lambda} - \alpha_1 d_1 - \frac{(\beta_1 - 1)r_1}{\alpha_1} + (n - r)p \frac{\frac{\partial R_1(T^*)}{\partial \alpha_1}}{R(T^*)} = 0 \quad (17)$$

$$-\frac{\alpha_1}{\lambda^2} - \frac{c_1 - r_1}{\lambda} = 0 \Rightarrow \lambda = \frac{\alpha_1}{c_1 - r_1} \quad (18)$$

Replacing λ by its formula (18) in equation (17), we get :

$$\frac{c_1 - \alpha_1 d_1 - \beta_1 r_1}{\alpha_1} + \sum_{i=1}^{r_1} \frac{\beta_1 t_i \left(\frac{t_i}{\alpha_1}\right)^{\beta_1-1}}{\alpha_1^2} + (n - r)p \frac{\frac{\partial R_1(T^*)}{\partial \alpha_1}}{R(T^*)} = 0 \quad (19)$$

Thus the optimal value for the parameter λ is $\frac{\alpha_1}{c_1 - r_1}$, so generating from the conditional distribution of α_1 is equivalent to generate from $G(c_1 - r_1, \frac{\alpha_1}{c_1 - r_1})$, using ARS method, where α_1 is obtained by solving equation (19)

3.1.3. The conditional distribution the scale parameter (α_2)

We do not write the term which does not involve α_2 in (10) . The conditional distribution of the scale parameter (α_2) becomes:

$$\pi(\alpha_2 | \beta_1, \beta_2, \alpha_1, p, t) \propto \alpha_2^{c_2 - r_2 - 1} \exp\left((\beta_2 - 1) \sum_{i=1}^{r_2} \ln\left(\frac{t_i}{\alpha_2}\right) - 2 \sum_{i=1}^{r_2} \ln\left(1 + \left(\frac{t_i}{\alpha_2}\right)^{\beta_2}\right) - \alpha_2 d_2 + (n - r) \ln(R(T^*))\right) \quad (20)$$

Since the conditional distribution of the scale parameter (α_2) given in (20), doesn't has any standard form we suggest a Gamma distribution $G(c_2 - r_2, \lambda)$ as a proposal(source) distribution. The parameter λ is chosen such that it gives the maximum acceptance rate Where the acceptance rate is given by:

$$AR(\alpha_2, \lambda) = \frac{\pi(\alpha_2 | \beta_1, \beta_2, \alpha_1, p, t)}{(c_2 - r_2, \lambda)}$$

using the same steps as before, we get $\frac{\alpha_2}{c_2 - r_2}$ is the optimal value for λ , which mean that α_2 can be generated from a gamma distribution $G(c_2 - r_2, \frac{\alpha_2}{c_2 - r_2})$ using ARS method, and α_2 is obtained by solving the following equation:

$$\frac{c_2 - \alpha_2 d_2 - \beta_2 r_2}{\alpha_2} + 2 \sum_{i=1}^{r_2} \frac{\beta_2 t_i (\frac{t_i}{\alpha_2})^{\beta_2-1}}{\alpha_2^2 ((\frac{t_i}{\alpha_2})^{\beta_2} + 1)} + (n-r)(1-p) \frac{\frac{\partial R_2(T^*)}{\partial \alpha_2}}{R(T^*)} = 0 \quad (21)$$

3.1.4. The conditional distribution the shape parameter (β_1)

We do not write terms that does not involve β_1 in (10).

The conditional distribution of the shape parameter (β_1) becomes:

$$\pi(\beta_1 | \beta_2, \alpha_1, \alpha_2, p, t) \propto \beta_1^{k_1+r_1-1} \exp(\beta_1 \sum_{i=1}^{r_1} \ln(\frac{t_i}{\alpha_1}) - \sum_{i=1}^{r_1} (\frac{t_i}{\alpha_1})^{\beta_1} - \beta_1 m_1 + (n-r) \ln(R(T^*))) \quad (22)$$

Using the same steps as before, we get $\lambda = \frac{\beta_1}{k_1+r_1}$ then β_1 can be generated using ARS method assuming a gamma distribution source $G(k_1 + r_1, \frac{\beta_1}{k_1+r_1})$ where β_1 is obtained by solving the following non-linear equation

$$-\sum_{i=1}^{r_1} (\frac{t_i}{\alpha_1})^{\beta_1} \log(\frac{t_i}{\alpha_1}) + \sum_{i=1}^{r_1} \log(\frac{t_i}{\alpha_1}) - m_1 + \frac{k_1+r_1}{\beta_1} + (n-r)p \frac{\frac{\partial R_1(T^*)}{\partial \beta_1}}{R(T^*)} = 0 \quad (23)$$

3.1.5. The conditional distribution the shape parameter (β_2)

We do not write terms that does not involve (β_2) in (10).

The conditional distribution of the shape parameter (β_2) becomes:

$$\pi(\beta_2 | \beta_1, \alpha_1, \alpha_2, p, t) \propto \beta_2^{k_2+r_2-1} \exp(\beta_2 \sum_{i=1}^{r_2} \ln(\frac{t_i}{\alpha_2}) - 2 \sum_{i=1}^{r_2} \ln(1 + (\frac{t_i}{\alpha_2})^{\beta_2}) - \beta_2 m_2 + (n-r) \ln(R(T^*))) \quad (24)$$

Using the same steps as before, we get $\lambda = \frac{\beta_2}{k_2+r_2}$ then β_2 can be generated using ARS method assuming a gamma distribution source $G(k_2 + r_2, \frac{\beta_2}{k_2+r_2})$ where β_2 is obtained by solving the following non-linear equation

$$-2 \sum_{i=1}^{r_2} \frac{(\frac{t_i}{\alpha_2})^{\beta_2} \log(\frac{t_i}{\alpha_2})}{(\frac{t_i}{\alpha_2})^{\beta_2} + 1} + \sum_{i=1}^{r_2} \log(\frac{t_i}{\alpha_2}) - m_2 + \frac{k_2+r_2}{\beta_2} + (n-r)(1-p) \frac{\frac{\partial R_2(T^*)}{\partial \beta_2}}{R(T^*)} = 0 \quad (25)$$

3.2. Bayesian estimation under type I censored non-postmortem data

for a test sample size of n items and for a number of sub-populations ' $C=2$ '. suppose that the test is stooped at a predetermined time " T^* ". At the end of the test we distinguish two types of data, those for failed items and the others for survived ones. Suppose that the switching labels of the failed items are unknowns, consequently the observations are unclassified and we can't assign the failed items to any one of the two sub-populations , thus the likelihood of entire sample is given by :

$$L = \prod_{i=1}^n (f(t_i))^{\sigma_i} (R(L_i))^{1-\sigma_i} \quad (26)$$

where indicator function σ_i is given by:

$$\sigma_i = \begin{cases} 1 & \text{if } t_i \leq l_i \\ 0 & \text{if } t_i > l_i \end{cases}$$

σ_i is the indicator of failure . If we assign to each item the couple (t_i, l_i) ; his exact lifetime is observed only if $t_i \leq l_i$,consequently ; $\sigma_i = 1$ if the failure time of the i^{th} item is exactly known and $\sigma_i = 0$ if it is censored. See [13]

Replacing the density function $f(t)$,and the reliability function $R(t)$ by theirs formulas given in (1) and 4 respectively in 26 we get :

$$L = \prod_{i=1}^n (p f_1(t_i) + (1-p) f_2(t_i))^{\sigma_i} (p R_1(l_i) + (1-p) R_2(l_i))^{1-\sigma_i} \quad (27)$$

The likelihood under type I censored non-postmortem data is complicated. For more simplification we formalize it using the notion of a missing data mechanism ([14]) where we define a missing data indicator matrix z such that:

$$z_{i,j} = \begin{cases} 1 & t_i \in (\text{component } j, j = 1,2) \Rightarrow t_i \text{ observed} \\ 0 & \text{otherwise} \Rightarrow t_i \text{ missing} \end{cases}$$

The likelihood function in this situation is given by :

$$L = \prod_{i=1}^n \prod_{j=1}^2 (p_j f_j^{\sigma_i}(t_i) R_j^{1-\sigma_i}(l_i))^{z_{i,j}} \quad (28)$$

Expanding (28) with respect to j for $j = 1,2$ we get :

$$L = \prod_{i=1}^n (p f_1^{\sigma_i}(t_i) R_1^{1-\sigma_i}(l_i))^{z_{i,1}} ((1-p) f_2^{\sigma_i}(t_i) R_2^{1-\sigma_i}(l_i))^{z_{i,2}} \quad (29)$$

Replacing in (29) we get :

$$L = \prod_{i=1}^n (p (e^{-\frac{t_i}{\alpha_1}})^{\beta_1})^{1-\sigma_i} \left(\frac{\beta_1 e^{-\frac{t_i}{\alpha_1}}}{\alpha_1} \left(\frac{t_i}{\alpha_1} \right)^{\beta_1-1} \right)^{\sigma_i} z_{i,1} ((1-p) \left(\frac{1}{\left(\frac{t_i}{\alpha_2} \right)^{\beta_2} + 1} \right)^{1-\sigma_i} \left(\frac{\beta_2 \left(\frac{t_i}{\alpha_2} \right)^{\beta_2-1}}{\alpha_2 \left(\left(\frac{t_i}{\alpha_2} \right)^{\beta_2} + 1 \right)^2} \right)^{\sigma_i} z_{i,2} \quad (30)$$

Simplifying (30) we get:

$$L = (1-p)^{n_2} p^{n_1} \left(\frac{\beta_1}{\alpha_1}\right)^{\hat{o}_1} \left(\frac{\beta_2}{\alpha_2}\right)^{\hat{o}_2} \exp \left((\beta_1 - 1) \sum_{i=1}^n \hat{v}_{i,1} \ln \left(\frac{t_i}{\alpha_1} \right) - \sum_{i=1}^n \left(\frac{t_i}{\alpha_1} \right)^{\beta_1} - \sum_{i=1}^n \hat{v}_{i,1} \left(\frac{t_i}{\alpha_1} \right)^{\beta_1} + (\beta_2 - 1) \sum_{i=1}^n \hat{v}_{i,2} \ln \left(\frac{t_i}{\alpha_2} \right) - 2 \sum_{i=1}^n \hat{v}_{i,2} \ln \left(\left(\frac{t_i}{\alpha_2} \right)^{\beta_2} + 1 \right) - \sum_{i=1}^n \hat{v}_{i,2} \ln \left(\left(\frac{t_i}{\alpha_2} \right)^{\beta_2} + 1 \right) \right) \quad (31)$$

Where

$\hat{v}_{i,j} = \sigma_i z_{i,j}$, $v_{i,j} = (1 - \sigma_i) z_{i,j}$, and $n_j = \sum_{i=1}^n z_{i,j}$, $\hat{o}_j = \sum_{i=1}^n \hat{v}_{i,j}$, $o_j = \sum_{i=1}^n v_{i,j}$. Combining the likelihood given by formula (31) with the prior given in (7) using Bayes' theorem, the joint posterior of ψ is given by:

$$\begin{aligned} \pi(\psi|t) \propto & \alpha_1^{c_1 - \hat{o}_1 - 1} \alpha_2^{c_2 - \hat{o}_2 - 1} \beta_1^{k_1 + \hat{o}_1 - 1} \beta_2^{k_2 + \hat{o}_2 - 1} (1-p)^{n_2 + s_2 - 1} p^{n_1 + s_1 - 1} \exp \left((\beta_1 - 1) \sum_{i=1}^n \hat{v}_{i,1} \ln \left(\frac{t_i}{\alpha_1} \right) \right. \\ & - \sum_{i=1}^n \left(\frac{t_i}{\alpha_1} \right)^{\beta_1} - \sum_{i=1}^n \hat{v}_{i,1} \left(\frac{t_i}{\alpha_1} \right)^{\beta_1} + (\beta_2 - 1) \sum_{i=1}^n \hat{v}_{i,2} \ln \left(\frac{t_i}{\alpha_2} \right) \\ & - 2 \sum_{i=1}^n \hat{v}_{i,2} \ln \left(\left(\frac{t_i}{\alpha_2} \right)^{\beta_2} + 1 \right) - \sum_{i=1}^n \hat{v}_{i,2} \ln \left(\left(\frac{t_i}{\alpha_2} \right)^{\beta_2} + 1 \right) - \alpha_1 d_1 - \alpha_2 d_2 - \beta_1 m_1 \\ & \left. - \beta_2 m_2 \right) \end{aligned} \quad (32)$$

The full conditional posterior PDFs for each unknown parameter are required for the implementation of the Gibbs sampling. Using (32) we derive the conditional posterior PDFs for each unknown parameter as follow:

3.2.1. The conditional distribution of the mixing proportion(p)

We do not write the term which does not involve p in (32).

The conditional distribution of the mixing proportion becomes:

$$\pi(p|\beta_1, \beta_2, \alpha_1, \alpha_2, t) \propto p^{n_1 + s_1 - 1} (1-p)^{n_2 + s_2 - 1} \quad (33)$$

Which is a beta distribution $\beta(n_1 + s_1, n_2 + s_2)$, thus the mixing proportion p can be generated using a beta function $\beta(n_1 + s_1, n_2 + s_2)$ assuming $n_i + s_i/i = \{1, 2\}$ are known. For practical purposes on OpenBugs and since the Dirichlet distribution is a generalization of the beta distribution. The mixing proportion can be generated using a "Dirichlet distribution" with parameters $n_1 + s_1, n_2 + s_2$

3.2.2. The conditional distribution of the shape parameter α_1

$$\pi(\alpha_1 | \beta_1, \beta_2, \alpha_2, p, t) \propto \alpha_1^{c_1 - \hat{o}_1 - 1} \exp \left((\beta_1 - 1) \sum_{i=1}^n \hat{v}_{i,1} \ln \left(\frac{t_i}{\alpha_1} \right) - \sum_{i=1}^n \left(\frac{t_i}{\alpha_1} \right)^{\beta_1} - \sum_{i=1}^n \hat{v}_{i,1} \left(\frac{t_i}{\alpha_1} \right)^{\beta_1} - \alpha_1 d_1 \right) \quad (34)$$

Since the conditional distribution of the scale parameter (α_1) given in (34), doesn't has any standard form we suggest the Gamma distribution $G(c_1 - \hat{o}_1, \lambda)$ as a proposal (source) distribution. The parameter λ is chosen such that it gives the maximum acceptance rate.

$$AR(\alpha_1, \lambda) = \frac{\pi(\alpha_1 | \beta_1, \beta_2, \alpha_2, p, t)}{\frac{e^{-\frac{\lambda}{\alpha_1}} \lambda^{c_1 - \hat{o}_1 - 1}}{\lambda^{c_1 - \hat{o}_1} \Gamma(c_1 - \hat{o}_1)}} \quad (35)$$

which result that the optimal value for λ is $\frac{\alpha_1}{c_1 - \hat{o}_1}$ and α_1 is obtained by solving the following non-linear equation:

$$\sum_{i=1}^n \frac{\beta_1 t_i \hat{v}_{i,1} \left(\frac{t_i}{\alpha_1} \right)^{\beta_1 - 1}}{\alpha_1^2} + (\beta_1 - 1) \sum_{i=1}^n -\frac{\hat{v}_{i,1}}{\alpha_1} + \sum_{i=1}^n \frac{\beta_1 t_i \left(\frac{t_i}{\alpha_1} \right)^{\beta_1 - 1}}{\alpha_1^2} + \frac{c_1 - \hat{o}_1}{\alpha_1} - d_1 = 0 \quad (36)$$

3.2.3. The conditional distribution the scale parameter (α_2)

We do not write the term which does not involve α_2 in (32). The conditional distribution of the scale parameter (α_2) becomes:

$$\begin{aligned} \pi(\alpha_2 | \beta_1, \beta_2, \alpha_1, p, t) &\propto \alpha_2^{c_2 - \hat{o}_2 - 2} \exp \left((\beta_2 - 1) \sum_{i=1}^n \hat{v}_{i,2} \ln \left(\frac{t}{\alpha_2} \right) \right. \\ &\quad \left. - 2 \sum_{i=1}^n \hat{v}_{i,2} \ln \left(\left(\frac{t}{\alpha_2} \right)^{\beta_2} + 1 \right) - \sum_{i=1}^n \hat{v}_{i,2} \ln \left(\left(\frac{t}{\alpha_2} \right)^{\beta_2} + 1 \right) - \alpha_2 d_2 \right) \end{aligned} \quad (37)$$

Since the conditional distribution of the scale parameter (α_2) given in (37), doesn't has any standard form we suggest the Gamma distribution $G(c_2 - \hat{o}_2, \lambda)$ as a source distribution. The parameter λ is chosen such that it gives the maximum acceptance rate.

$$AR(\alpha_2, \lambda) = \frac{\pi(\alpha_2 | \beta_1, \beta_2, \alpha_1, p, t)}{\frac{e^{-\frac{\lambda}{\alpha_2}} \lambda^{c_2 - \hat{o}_2 - 1}}{\lambda^{c_2 - \hat{o}_2} \Gamma(c_2 - \hat{o}_2)}} \quad (38)$$

which result that the optimal value for λ is $\frac{\alpha_2}{c_2 - \hat{o}_2}$ and α_2 is obtained by solving the following non-linear equation:

$$\sum_{i=1}^n \frac{\beta_2 t_i v_{i,2} (\frac{t_i}{\alpha_2})^{\beta_2-1}}{\alpha_2^2 ((\frac{t_i}{\alpha_2})^{\beta_2} + 1)} + 2 \sum_{i=1}^n \frac{\beta_2 t_i v_{i,2} (\frac{t_i}{\alpha_2})^{\beta_2-1}}{\alpha_2^2 ((\frac{t_i}{\alpha_2})^{\beta_2} + 1)} + (\beta_2 - 1) \sum_{i=1}^n -\frac{v_{i,2}}{\alpha_2} + \frac{c_2 - o_2}{\alpha_2} - d_2 = 0 \quad (39)$$

3.2.4. The conditional distribution the shape parameter (β_1)

We do not write terms that does not involve β_1 in (32).

The conditional distribution of the shape parameter (β_1) becomes:

$$\pi(\beta_1 | \beta_2, \alpha_1, \alpha_2, p, t) \propto \beta_1^{k_1 + o_1} \exp((\beta_1 - 1) \sum_{i=n}^n \frac{v_{i,1}}{\alpha_1} \ln(\frac{t_i}{\alpha_1}) - \sum_{i=n}^n (\frac{t_i}{\alpha_1})^{\beta_1} - \sum_{i=n}^n v_{i,1} (\frac{t_i}{\alpha_1})^{\beta_1} - \beta_1 m_1) \quad (40)$$

We suggest the use of a gamma distribution source $\text{Gamma}(k_1 + o_1, \lambda)$ for the parameter β_1 , which result that the optimal value for λ is $\frac{\beta_1}{k_1 + o_1}$ and β_1 is obtained by solving the following non-linear equation:

$$-\sum_{i=1}^n v_{i,1} (\frac{t_i}{\alpha_1})^{\beta_1} \log(\frac{t_i}{\alpha_1}) + \sum_{i=1}^n v_{i,1} \log(\frac{t_i}{\alpha_1}) - \sum_{i=n}^n (\frac{t_i}{\alpha_1})^{\beta_1} \log(\frac{t_i}{\alpha_1}) + \frac{k_1 + o_1}{\beta_1} - m_1 = 0 \quad (41)$$

3.2.5. The conditional distribution the shape parameter (β_2)

We do not write terms that does not involve β_2 in (32).

The conditional distribution of the shape parameter (β_2) becomes:

$$\pi(\beta_2 | \beta_1, \alpha_1, \alpha_2, p, t) \propto \beta_2^{k_2 + o_2 - 1} \exp((\beta_2 - 1) \sum_{i=n}^n \frac{v_{i,2}}{\alpha_2} \ln(\frac{t_i}{\alpha_2}) - 2 \sum_{i=n}^n \frac{v_{i,2}}{\alpha_2} \ln((\frac{t_i}{\alpha_2})^{\beta_2} + 1) - \sum_{i=n}^n v_{i,2} \ln((\frac{t_i}{\alpha_2})^{\beta_2} + 1) - \beta_2 m_2) \quad (42)$$

We suggest the use of a gamma distribution source $\text{Gamma}(k_2 + o_2, \lambda)$ for the parameter β_2 , which result that the optimal value for λ is $\frac{\beta_2}{k_2 + o_2}$ and β_2 is obtained by solving the following non-linear equation:

$$-\sum_{i=1}^n \frac{v_{i,2} (\frac{t_i}{\alpha_2}) \log(\frac{t_i}{\alpha_2})}{(\frac{t_i}{\alpha_2})^{\beta_2} + 1} - 2 \sum_{i=1}^n \frac{v_{i,2} (\frac{t_i}{\alpha_2})^{\beta_2} \log(\frac{t_i}{\alpha_2})}{(\frac{t_i}{\alpha_2})^{\beta_2} + 1} + \sum_{i=1}^n v_{i,2} \log(\frac{t_i}{\alpha_2}) + \frac{k_2 + o_2}{\beta_2} - m_2 = 0 \quad (43)$$

3.3. Numerical examples

Using the inverse CDF method in Mathcad 15 package, random samples of sizes $n = 1000$ and $n = 200$ were generated from the two components MWLLG with different values for the parameters. Termination points of the tests are selected in a way such that the censoring rate $C.R = (\frac{n-r}{n} \%)$ in the obtained samples is to be approximately 10 % and 30% respectively where r is the number of failed items. The maximum likelihood estimators are obtained using Modified Newton method with an allowed tolerance $\varepsilon = 10^{-5}$. Bayes estimators are obtained using OpenBugs software using 2000 iterations.

3.3.1. Type I censored postmortem data

Data are generated from the W-LLG mixture distributions with the following choice of parameters values : $(\alpha_1, \alpha_2) = (1,1), (\beta_1, \beta_2) = (1,3), p = 0.5$. The average estimates (A.E) and the associated mean squared errors (MSE) for model parameters are given in table (1)

Table1: The average estimates (A.E) and the associated mean squared errors (MSE) for the W-LLG mixture distributions under Type I censored post-mortem data

$(n,r) [CR,T^*]$		MLE	Bayes(Gibbs.S)		MSE			HPD interval	
			Prior 1	Prior 2	MLE	Prior 1	Prior 2	Prior 1	Prior 2
Case1: (200,180) [10%,2.19]	$\hat{\beta}_1$	1.1	0.767	0.738	0.022	0.056	0.07	[0.688,0.847]	[0.668,0.812]
	$\hat{\alpha}_1$	1.036	0.998	0.999	0.021	4.576×10^{-3}	1.113×10^{-3}	[0.880,1.138]	[0.942,1.074]
	$\hat{\beta}_2$	2.67	3.963	4.076	0.204	0.959	1.217	[3.630,4.312]	[3.630,4.606]
	$\hat{\alpha}_2$	0.911	0.980	1.003	0.013	3.291×10^{-3}	8.31×10^{-4}	[0.868,1.077]	[0.947,1.056]
	$\hat{p}$	0.499	0.497	0.499	1.71×10^{-3}	1.536×10^{-3}	1.526×10^{-3}	[0.492,0.503]	[0.494,0.505]
Case2: (200,140) [30%,1.22]	$\hat{\beta}_1$	0.953	0.749	0.730	0.016	0.064	0.074	[0.680,0.820]	[0.669,0.805]
	$\hat{\alpha}_1$	0.989	0.971	0.989	0.088	3.727×10^{-3}	1×10^{-3}	[0.869,1.078]	[0.9361,0.956]
	$\hat{\beta}_2$	3.161	4.043	4.125	0.201	1.127	1.388	[3.664,4.453]	[3.008,4.590]
	$\hat{\alpha}_2$	0.99	0.862	0.951	0.013	0.02	3.135×10^{-3}	[0.786,0.935]	[0.896,1.003]
	$\hat{p}$	0.481	0.480	0.476	7.12×10^{-3}	2.093×10^{-3}	2.289×10^{-3}	[0.475,0.486]	[0.471,0.482]
Case3: (1000,900) [10%,2.24]	$\hat{\beta}_1$	0.996	1.028	1.001	1.891×10^{-3}	0.017	1.005×10^{-3}	[0.769,1.275]	[0.940,1.061]
	$\hat{\alpha}_1$	1.033	0.956	1.000	5.463×10^{-3}	0.016	8.399×10^{-3}	[0.718,1.187]	[0.944,1.052]
	$\hat{\beta}_2$	2.923	3.005	2.995	0.026	7.445×10^{-3}	9.917×10^{-4}	[2.844,3.181]	[2.809,3.188]
	$\hat{\alpha}_2$	0.983	0.982	1.01	1.219×10^{-3}	6.059×10^{-3}	9.561×10^{-4}	[0.819,1.123]	[0.953,1.062]
	$\hat{p}$	0.496	0.494	0.495	3.371×10^{-4}	3.181×10^{-4}	2.834×10^{-4}	[0.493,0.496]	[0.493,0.495]
Case4: (1000,700) [30%,1.26]	$\hat{\beta}_1$	0.91	0.989	0.988	0.01	0.013	0.013	[0.772,1.214]	[0.915,1.050]
	$\hat{\alpha}_1$	1.2	0.946	0.964	0.051	0.014	0.012	[0.742,1.155]	[0.906,1.027]
	$\hat{\beta}_2$	3.373	2.997	2.885	0.018	9.832×10^{-3}	0.023	[2.798,3.179]	[2.680,3.100]
	$\hat{\alpha}_2$	0.911	0.969	0.824	9.256×10^{-3}	7.282×10^{-3}	0.037	[0.817,1.123]	[0.776,0.873]
	$\hat{p}$	0.558	0.489	0.488	3.94×10^{-3}	5.057×10^{-4}	5.124×10^{-4}	[0.487,0.49]	[0.487,0.489]

3.3.2. Type I censored non postmortem data:

Data are generated from the W-LLG mixture distributions with the following choice of parameters values : $(\alpha_1, \alpha_2) = (1.5, 0.5), (\beta_1, \beta_2) = (2, 2.5), p = 0.3$. The average estimates and the associated mean squared errors (MSE) for α_1, β_1 are given in table (2)

Table2 : The average estimates (A.E) and the associated mean squared errors (MSE) for the W-LLG mixture distributions under Type I censored non post-mortem data

(n,r) [CR,T*]		MLE	Bayes(Gibbs.S)		MSE			HPD interval	
			Prior 1	Prior 2	MLE	Prior 1	Prior 2	Prior 1	Prior 2
Case 1: (200,180) [10%,1.79]	$\hat{\beta}_1$	2.518	2.152	2.159	9.842	0.033	0.038	[1.988,2.365]	[1.944,2.378]
	$\hat{\alpha}_1$	1.708	1.162	1.316	0.434	0.122	0.064	[0.985,1.318]	[1.082,1.746]
	$\hat{\beta}_2$	2.633	3.492	4.412	0.194	1.01	3.881	[3.198,3.808]	[2.937,4.975]
	$\hat{\alpha}_2$	0.501	0.419	0.430	0.011	6.62×10^{-3}	5.174×10^{-3}	[0.394,0.447]	[0.396,0.462]
	$\hat{p}$	0.229	0.301	0.305	0.098	6.159×10^{-4}	6.907×10^{-4}	[0.2560,354]	[0.255,0.356]
Case 2: (200,140) [30%,0.940]	$\hat{\beta}_1$	3.275	2.153	2.132	120.05	0.035	0.025	[1.9512,357]	[1.964,2.293]
	$\hat{\alpha}_1$	1.328	1.198	1.204	0.183	0.1	0.102	[1.051,1.428]	[0.999,1.505]
	$\hat{\beta}_2$	2.701	3.255	4.152	0.25	0.602	2.855	[2.861,3.575]	[3.209,4.814]
	$\hat{\alpha}_2$	0.476	0.419	0.443	0.025	6.737×10^{-3}	3.469×10^{-3}	[0.392,0.444]	[0.4110,473]
	$\hat{p}$	0.279	0.298	0.305	0.286	6.39×10^{-4}	6.783×10^{-4}	[0.250,0.347]	[0.256,0.354]
Case 3 : (1000,900) [10%,1.86]	$\hat{\beta}_1$	1.916	2.009	1.988	0.461	5.551	5.249	[1.867,2.155]	[1.839,2.111]
	$\hat{\alpha}_1$	1.594	1.508	1.41	0.104	0.017×10^{-3}	0.014×10^{-3}	[1.254,1.770]	[1.125,1.648]
	$\hat{\beta}_2$	2.597	2.693	2.74	0.07	0.042	0.076	[2.557,2.834]	[2.595,2.893]
	$\hat{\alpha}_2$	0.508	0.456	0.436	2.11×10^{-3}	2.237×10^{-3}	4.458×10^{-3}	[0.423,0.494]	[0.404,0.472]
	$\hat{p}$	0.294	0.299	0.301	0.028	1.398×10^{-4}	1.367×10^{-4}	[0.278,0.324]	[0.278,0.324]
Case 04: (1000,700) [30%,0.942]	$\hat{\beta}_1$	1.697	1.961	1.943	0.132	6.05×10^{-3}	8.007×10^{-3}	[1.830,2.091]	[1.816,2.079]
	$\hat{\alpha}_1$	1.192	1.514	1.468	0.139	0.016	0.032	[1.317,1.823]	[1.155,1.816]
	$\hat{\beta}_2$	2.989	2.399	2.386	0.435	0.015	0.018	[2.273,2.522]	[2.257,2.524]
	$\hat{\alpha}_2$	0.446	0.4791	0.468	5.77×10^{-3}	7.462×10^{-3}	1.539×10^{-4}	[0.448,0.516]	[0.428,0.512]
	$\hat{p}$	0.491	0.299	0.302	0.067	1.527×10^{-4}	1.516×10^{-4}	[0.272,0.321]	[0.279,0.325]

It was concluded that:

- Maximum likelihood and Bayes estimators are almost the same as the real values.
- For a fixed censoring rate (C.R) and by increasing the sample size, we get better estimates and smaller MSE's.
- For a fixed sample size (n) and by decreasing the censoring rate (C.R), we get better estimates and smaller MSE's. This is due to the loss of information caused by the censored data.
- As the sample size increase and the censoring rate decrease, as we get batter estimates, smaller MSE's.
- The choice of a convenient prior improves the quality of estimators which means that the available priory knowledge on parameters can improve the quality of estimators.
- The Gibbs sampling method produce better estimates than the Maximum likelihood method especially when the sample size (n) is large.

4. FITTING REAL DATA

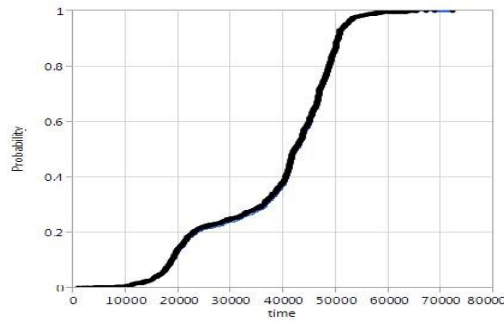
4.1. Backblaze hard drive data

BackBlaze is a cloud storage firm. Since 2013 it has published raw performance data on hard drives operating at its data centres. The data are publicly available through their website (<https://www.backblaze.com/b2/hard-drive-test-data.html>, Accessed January 1, 2018).

Every day Backblaze records the SMART stats reported by the hard drives that are spinning continuously in their data centres. 24 885 hard drives have been running since 2013.

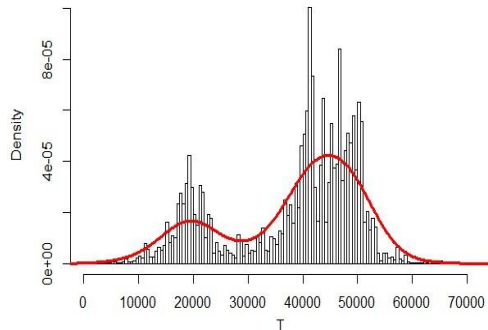
Figure (1) shows the Kaplan-Meier (K-M) estimate for the failure function. The curve of the failure function increases slowly till about 10.000 hours (early failures), then it rises more rapidly till about 50.000 hours, then it returned to increase slowly till about 70.000 with (wear-out failures) which means that there is more than one failure mode;

Figure 1: K-M estimate for the failure function.



From graph (2) it is clear also that there is more than one mode of failure, which represents a practical justification to use mixture modelling if we assume that each failure mode has its own probability distribution function. In addition and according to [15], [16] and [17] computer hard drives are among products that have a mixture of infant and Wear-out Failures which represent a theoretical justification to use mixture distributions.

Figure 2: Histogram of H.drives failure Data with Kernel Density Plot.



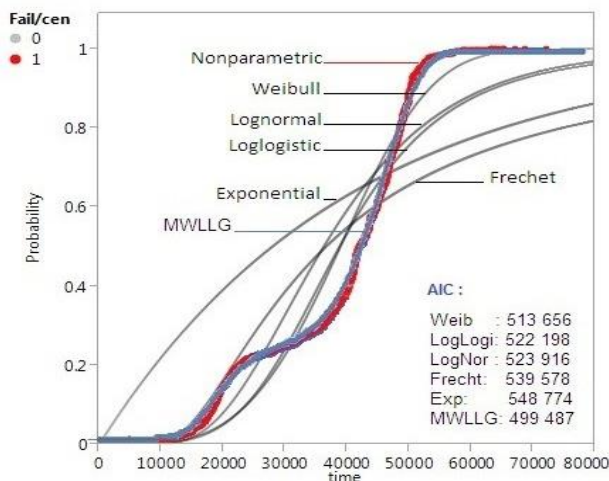
Fitting the data using univariate lifemodells (Weibull, Log-Logistic, Exponential, LogNormal, Frechet) shows that the Weibull distribution present the best fit for the data with the smallest "AIC", The log-logistic distribution model present a good substitute for Weibull distribution.

Next we fit the data to the MWLLG. Parameters estimates and relatives AIC and BIC for MWLLG vs its components models are given in table (3).

Table 3: Fitting Hard drives failure data to the MWLLG VS its component models

Parameters	MLE MWLLG	Bayes (Gibbs Sampling)		MLE single component	
		prior 1	prior 2	Weibull	Log Logistic
β_1	9.313	9.323	9.283	3.823	--
α_1	46959.882	46840.00	46910.00	42807.13	--
β_2	5.977	5.792	5.991	--	4.460719
α_2	19896.751	19.850.00	19870.00	--	38739.25
P	0.723	0.740	0.732	--	--
AIC	499486.923	499392.095	499351.041	513656.38	522198.00
BIC	499527.532	499432.705	499391.651	513672.62	522214.35

Figure 3: Goodness-of-fit CDF plots.



The maximum likelihood estimators are obtained using the Modified Newton method with an allowed tolerance $\varepsilon = 10^{-5}$. From estimation It was concluded that:

- The Bayes estimators under Gibbs sampling are almost the same as the starting values (MLE's).
- The Gibbs sampling method under some appropriate priors (prior 2) produce smaller "AIC" and "BIC" comparing to the ML method.
- Fitting the HDD data to the Mixture of W-LLG gives smaller "AIC's" than fitting to the Weibull distribution or the log logistic distribution. This mean that the MWLLG model provide more flexibility than its single component distributions. This flexibility is clears in figure 3 where the curve of the theoretical distribution (MWLLG) adequate the curve of the empirical distribution.

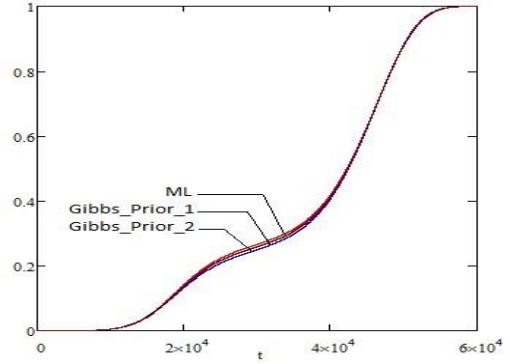
4.2. The reliability and the failure rate function

$R(\hat{t})$ and $h(\hat{t})$ are obtained by replacing the unknown's parameters by their corresponding estimated values given in table (3).

4.2.1. The reliability function

Figure (4) shows the estimated reliability function for HDD data, where it is clear that the curves of the CDF functions under the ML method and the Gibbs sampling method are almost the same except a small difference during the useful life period. It is also clear that the probability that a hard drive is still running during the useful life period is small under the Gibbs sampling method than for the ML method.

Figure 4: The reliability function of HDD failure data

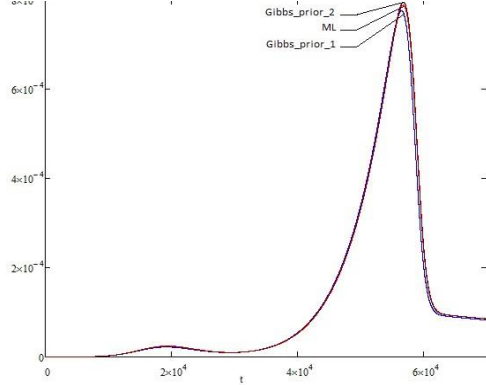


4.2.2. The failure rate function

Figure (5) shows that the failure rate function of hard drives is "M-shaped". Note that fewer distributions are capable of modelling more complicated shapes such the M-shaped failure rate.

Note that the hard drives are studied in a large interval of time [0,72153 hours], this can depict all regions of the failure rate. In addition, the sample used in our research contains failure information about four (04) hard drive models (Toshiba, Western digital, Seagate, HGST) with different storage sizes, which means a variety of failure causes and consequently a complex failure rate shape.

Figure 5: The Failure rate function of HDD failure data.



5. CONCLUSIONS

In this paper the statistical inference of the unknown parameters of the W-LLg mixture distributions under Type-I censoring has been considered. We consider the Bayes estimates of the unknown parameters based on squared error loss function, and it is observed that they cannot be obtained explicitly. Gibbs sampling technique has been used to obtain Bayes estimates and it is observed that the implementation of the Gibbs sampling procedure is very simple in this case. The estimations of parameters are evaluated under various censoring rates and under different priors. Finally, application of the model to hard drives failure data, where it is shown that the MWLLG model provide more flexibility than its component distribution models in fitting this data.

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