

AN EVOLUTIONARY GAME FOR ALGERIAN TELECOMMUNICATION SERVICE PROVIDERS

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Abstract: This article recognizes that in Algerian telecommunication market in which revenue optimization is attempted, an actual demand curve and its parameters are generally unobservable.

Herein, we describe the dynamics of demand as a continuous time differential equation based on an evolutionary game theory perspective. We then observe realized sales data to obtain estimates of parameters that govern the evolution of demand; these are refined on a discrete time scale.

The resulting model takes the form of a differential variational inequality. We present an algorithm based on a gap function for the differential variational inequality and report its numerical performance for an application on Algerian telecommunication service providers (ATSP).

Keywords: ATSP, Pricing, Differential games, Kalman filters.

Résumé: Cet article admet que dans le marché algérien des télécommunications dans lequel une optimisation des revenus est tentée, une courbe de la demande réelle et ses paramètres ne sont généralement pas observables.

Ici, nous décrivons la dynamique de la demande comme une équation différentielle en temps continu basé sur une perspective de la théorie des jeux évolutionnaires. On observe alors les données de ventes réalisées pour obtenir des estimations des paramètres qui régissent l'évolution de la demande, ces derniers sont raffinés sur une échelle de temps discret.

Le modèle qui en résulte prend la forme d'une inégalité différentielle de variation. Nous présentons un algorithme basé sur une fonction de l'écart de l'inégalité différentielle de variation et démontrons sa performance numérique pour une application sur les fournisseurs de services de télécommunication algérien (FSTA).

Mots clés: FSTA, Pricing, Jeux différentiels, Filtre de Kalman.

1. INTRODUCTION

One of the most challenging tasks for Algerian telecommunication service providers (ATSP) is the definition of a model for management that yields a constant satisfaction of profits maximization rule. In Algeria, telecommunication

firms' evolution dazzles from strict public regulations rather than competitive development. There is albeit three ATSP in this emerging market representing a small-sized oligopoly. Therefore, ATSP are competing in an oligopolistic setting, each with the objective of maximizing their profits. ATSP have very high fixed costs compared to their relatively low variable or operating costs. Therefore, the providers focus only on maximizing their own revenue. Each company provides a set of services for which each service type is assumed to be homogeneous. For example, the difference between a 1 minute call on DJEEZY and a 1 minute call on NEDJMA is indiscernible by customers; the only differences that the customers perceive are the prices charged by the different ATSP. All ATSP can set the price for each of their services. The price that they charge for each service in one time period will affect the demand that they receive for that service in the next time period. The price that ATSP charges is compared to the rolling average price of their competitors. The amount of service that each company can provide has an upper bound due to capacity constraints. The providers must therefore choose prices that create an amount of demand for their services that will maximize their revenue while ensuring that the demands do not exceed their capacities.

One of the most important issues is how to model ATSP demand learning. Demand is usually represented as a function of price (McGill and van Ryzin, 1999; Talluri and van Ryzin, 2004), explicitly and/or implicitly, and the root tactic is to change prices dynamically to maximize immediate or short-run revenue. In this sense, the more accurate the model of demand employed in optimization, the more revenue we can generate. Although demand may be viewed theoretically as the result of utility maximization, an actual demand curve and its parameters are generally unobservable in most markets.

ATSP are in oligopolistic game-theoretic competition according to a learning process that is similar to evolutionary game-theoretic dynamics and for which price changes are proportional to their signed excursion from a market clearing price. It is stressed that in this model ATSP are setting prices for their services while simultaneously determining the levels of demand they will serve. This is unusual in that, typically, firms in oligopolistic competition are modeled as setting either prices or output flows. The joint adjustment of prices and outputs is modeled here by comparing current price to the price that would have cleared the market for the demand that has most recently been served. However, ATSP are unable to make this comparison until the current round of play is completed as knowledge of the total demand served by all competitors is required. Kachani et al. (2004) put forward a model for service providers to address such joint pricing and demand learning in an oligopolistic setting with fixed capacity constraints. The model they consider assumes the demand faced by service provider is a linear function of its price and other competitors' prices; each company learns to set their parameters over time, although the impact of a

change in price on demand in one period does not automatically propagate to latter time periods. In the current paper it is allowed this impact to propagate to all subsequent time periods.

In this paper, we will only be considering a single class of customers, so-called bargain-hunting buyers, searching for telecommunication services at the most competitive prices; these buyers are willing to sacrifice some convenience for the sake of a lower price. Because the services are assumed to be homogeneous, if two ATSP offer the same price, the tie is broken randomly. In other words, the consumer has no concept of brand preference. We first describe the dynamics of demand as a differential variational inequality with Kalman filtering process based on non-zero sum evolutionary game among ATSP then simulate actual sales data to obtain estimates of parameters that govern the evolution of demand.

2. LITERATURE REVIEW

Differential variational inequalities, or DVIs, are infinite-dimensional variational inequalities constrained by ordinary differential equations, called state dynamics in optimal control theory. DVIs arise in mechanics, mathematical economics, transportation research, and many other complex engineering systems. Pang and Stewart (2008) defined DVIs formally and present illustrative applications to differential Nash games and rigid-body dynamics. Moreover, Friesz et al. (2006) used a DVI to model shippers' oligopolistic competition on networks. The DVI of interest here should not be confused with the differential variational inequality of Aubin and Cellina (1984), which may be called a variational inequality of evolution (Pang and Stewart, 2008).

The non-cooperative dynamic oligopolistic competition among service providing agents may be placed in the form of a dynamic variational inequality wherein each individual agent develops for itself a distribution plan that is based on current knowledge of non-own price patterns. Each such agent is, therefore, described by an optimal control problem for which the control variables are the prices of its various service classes; this optimal control problem is constrained by dynamics that describe how the agent's share of demand varies with time. The Cournot-Nash assumption that game agents are non-cooperative allows the optimality conditions of individual agents to be combined to obtain a single DVI.

Forecasting demand is crucial in pricing and planning for any firm in that the forecasts have huge impacts on the revenues. In revenue management demand is usually modeled as an exogenous stochastic process with a known distribution (Gallego and van Ryzin, 1994; Feng and Gallego, 1995). Such models are restrictive because (1) they depend largely on complete knowledge of demand before pricing starts; (2) they do not incorporate any learning mechanisms

which will improve the demand estimation as more information becomes available.

In this paper, demands for each ATSP are governed by dynamics controlled by prices. However, the parameters in the demand dynamics are unknown. Demand forecasting is often flexible in the sense that it is able to handle incomplete information about the demand function. Furthermore, demand is learned over time and each ATSP can update demand functions as new information becomes available. By using a learning mechanism, ATSP can better estimate the demand functions, thereby improving its profitability.

Demand learning has been studied extensively in many research areas. A typical approach to model learning is based on the notion of Bayesian learning. In this last, demand at any time is a stochastic process following certain distribution with unknown parameters. Observed demand data are used to modify beliefs about unknown parameters based on Bayes' rule. In this approach, uncertainty in the parameter is resolved as more observations become available, and the distribution of any demand will approach its true distribution (Murray and Silver, 1966; Epen and Iyer, 1997; Bitran and Wadhwa, 1996). Recently, researchers have developed other learning mechanisms to resolve demand uncertainty. Berstimas and Parakis (2006) develop a demand learning mechanism which estimates the parameters for the linear demand function via a least square method. Yelland and Lee (2003) use class II mixture models to capture both the uncertainty in model specification and demand. They demonstrate that class II mixture models are more efficient in forecasting demand. Lin (2006) proposes to use real time demand data to improve the estimation of customer arrival rates which in turn can be used to better predict future demand distributions and develops a variable-rate policy which is immune to the changes in the customer arrival rate.

The demand learning approach proposed in this article is based on the Kalman filter. As a state-space estimation method, Kalman filtering has gained attention in economics as one of the most successful forecasting methods. The Kalman filter was developed by Kalman (1960) and Kalman and Bucy (1961), originally to filter out system and sensor noise in electrical/mechanical systems, is based on a system of linear state dynamics, observation equations and normally distributed noise terms with mean zero. The Kalman filter provides an a priori estimate for the model parameter, which is adjusted to a posteriori estimate by combining the observations. With this new model parameter, one repeatedly solves the dynamic pricing problem to maximize revenue in the next time period. For a detailed discussion and derivations of Kalman filter dynamics, see Bryson and Ho (1975), or, for an introduction suitable for revenue management researchers, see Talluri and van Ryzin (2004).

In the economics literature, Balvers and Cosimano (1990) study a stochastic linear demand model in a dynamic pricing problem and obtain a dynamic programming formulation to maximize revenue. They use the Kalman filter to estimate the exact value of the intercept and elasticity in the stochastic linear demand model. Closely related, Carvalho and Puterman (submitted for publication) consider a log-linear demand model whose parameters are also stochastic, and test the model using Monte Carlo simulation. In addition, Xie et al. (1997) find an application of the Kalman filter in estimation of new product diffusion models. They concluded the Kalman filter approach gives superior predictions compared to other estimation methods in such an environment.

Among available algorithms to solve the variational inequality problems, gap functions allow a variational inequality problem to be converted to an equivalent optimization problem, whose objective function is always non-negative and whose optimal objective function value is zero if and only if the optimal solution solves the original variational inequality problem. A number of algorithms in this class have been developed for finite-dimensional variational inequalities; see for example, Zhu and Marcotte (1994) and Yamashita et al. (1997). For infinite-dimensional problems, Zhu and Marcotte (1998) and Konnov et al. (2002) provide descent methods using gap functions for Banach spaces and Hilbert spaces, respectively.

However, only a few numerical schemes for the DVIs have been reported in the literature despite of the increasing importance of the DVIs applied to differential game theory. Pang and Stewart (2008) suggest two algorithms, namely, the time stepping method and the sequential linearization method. The former discretizes the time horizon with finite differences for the state dynamics, yielding finite-dimensional variational inequalities. To ensure the convergence of the time stepping method, one needs linearity with respect to control in the state dynamics. To overcome this limitation, the later scheme, the sequential linearization method, sequentially approximates the given DVI by a sub-DVI for each state and control variable, which is solved by available algorithms such as the time stepping method. In this paper, we employ a descent method based on gap functions devised by Kwon and Friesz (2007) for infinite-dimensional variational inequalities to obtain an equivalent optimal control problem.

The finite or infinite-dimensional variational inequality problem (VIP) is, for a compact and convex set U and function F , to find $u^* \in U$ such that

$$\langle F(u^*), u - u^* \rangle \geq 0 \quad \forall u \in U$$

Where $\langle \cdot, \cdot \rangle$ denotes the corresponding inner product. It is well known that a VIP is closely related to an optimization problem. We will see then a differential variational inequality problem is closely related to an optimal control problem. We begin by letting

$$\begin{aligned} u \in (L^2[t_0, t_f])^m, x(u, t) = \arg \left\{ \frac{dy}{dt} = f(y, u, t), y(t_0) = y, \Gamma[y(t_f), t_f] = 0 \right\} \\ \in (\mathcal{H}^1[t_0, t_f])^n \end{aligned} \quad (1)$$

The entity $x(u, t)$ should be interpreted as an operator that tells us the state variable x for each vector u and each time $t \in [t_0, t_f] \subset \mathbb{R}_+^1$; constraints on u are enforced separately. We assume that every control vector is constrained to lie in a set U , where U is defined to ensure the terminal conditions may be reached from the initial conditions intrinsic to (1). In light of equation (1), the variational inequality of interest to us takes the form:

Find $u^* \in U$ such that

$$\langle F(x(u^*, t), u^*, t), u - u^* \rangle \geq 0 \quad \forall u \in U \quad (2)$$

Where $U \subseteq (L^2[t_0, t_f])^m$

$$x^0 \in \mathbb{R}^n$$

$$F: (\mathcal{H}^1[t_0, t_f])^n \times (L^2[t_0, t_f])^m \times \mathbb{R}_+^1 \rightarrow (L^2[t_0, t_f])^m$$

$$f: (\mathcal{H}^1[t_0, t_f])^n \times (L^2[t_0, t_f])^m \times \mathbb{R}_+^1 \rightarrow (L^2[t_0, t_f])^n$$

$$\Gamma: (\mathcal{H}^1[t_0, t_f])^n \times \mathbb{R}_+^1 \rightarrow (\mathcal{H}^1[t_0, t_f])^r$$

Note that $(L^2[t_0, t_f])^m$ is the m -fold product of the space of square-integrable functions $L^2[t_0, t_f]$ and the inner product in (2) is defined by

$$\langle F(x(u^*, t), u^*, t), u - u^* \rangle \equiv \int_{t_0}^{t_f} [F(x(u^*, t), u^*, t)]^T (u - u^*) \geq 0$$

While $(\mathcal{H}^1[t_0, t_f])^n$ is the n -fold product of the Sobolev space $\mathcal{H}^1[t_0, t_f]$. We refer to (2) as a differential variational inequality and give it the symbolic name $DVI(F, f, U)$.

To analyze (2) we will rely on the following notion of regularity:

Definition 1 (Regularity). We call $DVI(F, f, U)$ regular if:

1. $x(u, t): (L^2[t_0, t_f])^m \times \mathbb{R}_+^1 \rightarrow (\mathcal{H}^1[t_0, t_f])^n$ exists and is continuous and Gateaux-differentiable with respect to u ;¹
2. $\Gamma(x, t)$ is continuously differentiable with respect to x ;
3. $F(x, u, t)$ is continuous with respect to x and u ;

¹This condition is guaranteed when the other regularity conditions are satisfied. See Bressan and Piccoli (2005).

4. $f(x, u, t)$ is convex and continuously differentiable with respect to x and u ;
5. U is convex and compact; and
6. $x^0 \in \mathbb{R}^n$ is known and fixed.

The motivation for this definition of regularity is to parallel as closely as possible those assumptions needed to analyze traditional optimal control problems from the point of view of infinite dimensional mathematical programming.

Furthermore, there is a fixed point form of $DVI(F, f, U)$. In particular we state the following result without proof (see Mookherjee, 2006):

Theorem of Fixed Point Problem. *When regularity in the sense of Definition 1 holds, $DVI(F, f, U)$ is equivalent to the following fixed point problem:*

$$u = P_U[u - \alpha F(x(u), u, t)]$$

Where $P_U[\cdot]$ is the minimum norm projection onto $U \subseteq (L^2[t_0, t_f])^m$ and $\alpha \in \mathbb{R}_{++}^1$.

We now state the following existence result (for proof see Mookherjee (2006)):

Theorem of existence. *When the regularity in the sense of Definition 1 holds, $DVI(F, f, U)$ has a solution.*

3. MODELING THE GAME

As a basic notation, we denote the set of ATSP as $\mathcal{F}$, each of whom is providing a set of services $\mathcal{S}$. Continuous time is denoted as the scalar $t \in \mathbb{R}_+^1$, while t_0 is the finite initial time and $t_f \in \mathbb{R}_{++}^1$ is the finite terminal time so that $t \in [t_0, t_f] \subset \mathbb{R}_+^1$.

Each firm $f \in \mathcal{F}$ controls price

$$\pi_i^f \in L^2[t_0, t_f]$$

Corresponding to each service type $i \in \mathcal{S}$, where $L^2[t_0, t_f]$ is the space of square-integrable functions for the real time interval $[t_0, t_f] \subset \mathbb{R}_+^1$. The control vector of each firm $f \in \mathcal{F}$ is

$$\pi^f \in (L^2[t_0, t_f])^{|\mathcal{S}|}$$

Which is the concatenation of

$$\pi \in (L^2[t_0, t_f])^{|\mathcal{S}| \times |\mathcal{F}|}$$

The complete vector of controls. We also let

$$D_i^f(\pi, t): (L^2[t_0, t_f])^{|\mathcal{S}| \times |\mathcal{F}|} \times \mathbb{R}_+^1 \rightarrow \mathcal{H}^1[t_0, t_f]$$

Denote the demand for service $i \in \mathcal{S}$ of firm $f \in \mathcal{F}$ and define the vector of all such demands for firm f to be

$$D^f \in (\mathcal{H}^1[t_0, t_f])^{|\mathcal{S}|}$$

All such demands for service $i \in \mathcal{S}$ of firm $f \in \mathcal{F}$ are denoted by

$$D \in (\mathcal{H}^1[t_0, t_f])^{|\mathcal{S}| \times |\mathcal{F}|}$$

We will use the notation

$$\pi^{-f} = (\pi_i^g: i \in \mathcal{S}, g \in \mathcal{F} - \{f\})$$

For the vector of all service levels provided by the competitors of firm $f \in \mathcal{F}$.

In evolutionary game theory the notion of comparing a moving average to the current state is used to develop ordinary differential equations describing learning processes (Fudenberg and Levine, 1999). To proceed, we first assume the qualities of services provided by different ATSP are homogeneous; hence the customers' decisions depend only on their prices.

The demand for the service offerings of firm $f \in \mathcal{F}$ evolve according to the following evolutionary game-theoretic dynamics:

$$\frac{dD_i^f}{dt} = \eta_i^f (\tilde{\pi}_i - \pi_i^f) \quad \forall i \in \mathcal{S}, f \in \mathcal{F} \quad (3)$$

$$D_i^f(t_0) = K_{i,0}^f \quad \forall i \in \mathcal{S}, f \in \mathcal{F} \quad (4)$$

Where $\tilde{\pi}_i$ is the moving average price for service $i \in \mathcal{S}$ given by

$$\tilde{\pi}_i(t) = \frac{1}{|\mathcal{F}|(t - t_0)} \int_{t_0}^t \sum_{g \in \mathcal{F} - \{f\}} \pi_i^g(\tau) d\tau \quad \forall i \in \mathcal{S}$$

While $K_{i,0}^f \in \mathbb{R}_{++}^1$ and $\eta_i^f \in \mathbb{R}_{++}^1$ are exogenous parameters for each $i \in \mathcal{S}$ and $f \in \mathcal{F}$. The ATSP set the parameter η_i^f by analyzing the past demand data and the sensitivity of the demand with respect to price. The demand for service type i of a firm f changes over time in accordance with the excess between the firm's price and the moving average of all agents' prices for the particular service. The coefficient η_i^f controls how quickly demand reacts to price changes for each firm f and service type i . Some ATSP may specialize in certain services and may be able to adjust more quickly than their competitors.

In the literature, it is commonly assumed that observed demands in periods t and $t + 1$ are independent and demand at time t is influenced by price at time t (which is set at time $t - 1$). In our model, we captured this “learning” behavior in a naïve way: customers have some “reference price” of a differentiated service which gets updates at the end of every period as they learn about the market condition and demand for service is proportional to the price differential (tatonnement dynamics). This way demand at time t not only depends on price set at time t , but on complete price trajectories $[0, t]$. In this sense, the dynamic equation (3) based on “moving average price” of demand learning is more realistic. The “replicator dynamics” (Hofbauer and Sigmund, 1998), of which equation (3) is an instance, reflects a widely respect theoretical view or hypothesis that learning is the notion of comparison to the moving average.

These dynamics represent a learning mechanism for the ATSP. As stated here, the dynamics are reminiscent of replicator dynamics which are used in evolutionary games. The rate of growth of demand, can be viewed as the rate of growth of the firm f with respect to service type i . This growth follows the “basic tenet of Darwinism” and may be interpreted as the difference between the fitness (price) of the firm for the service and the rolling average fitness of all the agents for that service.

There are positive upper and lower bounds, based on market regulations or knowledge of customer behavior, on service prices charged by ATSP. Thus, we write

$$\pi_{min,i}^f \leq \pi_i^f \leq \pi_{max,i}^f \quad \forall i \in S, f \in \mathcal{F}$$

Where the $\pi_{min,i}^f \in \mathbb{R}_{++}^1$ and $\pi_{max,i}^f \in \mathbb{R}_{++}^1$ are known constants. Similarly, there will be a lower bound on the demand for services of each type by each firm as negative demand levels are meaningless; that is

$$D_i^f \geq 0 \quad \forall i \in S, f \in \mathcal{F}$$

Let $\mathcal{R}$ be the set of resources that ATSP can utilize to provide the services, while $|\mathcal{R}|$ is the cardinality of $\mathcal{R}$. Define the incidence matrix (Talluri and van Ryzin, 2004) $A = (a_{lm})$ by

$$a_{lm} = \begin{cases} 1 & \text{if resource } l \text{ is used by the service type } m \\ 0 & \text{otherwise} \end{cases}$$

Joint resource-constraints for firm f are

$$0 \leq AD_i^f \leq C_i^f \quad \forall i \in S, f \in \mathcal{F} \quad (5)$$

Where C_i^f is firm f 's capacity of resources used for service i .

Since ATSP have negligible variable costs and high fixed costs, each firm's objective is to maximize revenue which in turn ensures the maximum profit. We further note that each firm $f \in \mathcal{F}$ faces the following problem: with π^{-f} as exogenous inputs, solve the following optimal control problem:

$$\max_{\pi^f} J_f(\pi^f, \pi^{-f}, t) = \int_{t_0}^{t_f} e^{-\rho t} \left(\sum_{i \in \mathcal{S}} \pi_i^f D_i^f \right) dt - e^{-\rho t_0} \varphi_0^f \quad (6)$$

Subject to

$$\frac{dD_i^f}{dt} = \eta_i^f (\tilde{\pi}_i - \pi_i^f) \quad \forall i \in \mathcal{S}, f \in \mathcal{F} \quad (7)$$

$$D_i^f(t_0) = K_{i,0}^f \quad \forall i \in \mathcal{S}, f \in \mathcal{F} \quad (8)$$

$$\pi_{\min,i}^f \leq \pi_i^f \leq \pi_{\max,i}^f \quad \forall i \in \mathcal{S}, f \in \mathcal{F} \quad (9)$$

$$D_i^f \geq 0 \quad \forall i \in \mathcal{S}, f \in \mathcal{F} \quad (10)$$

$$0 \leq AD_i^f \leq C_i^f \quad \forall i \in \mathcal{S}, f \in \mathcal{F} \quad (11)$$

Where φ_0^f is the fixed cost for firm f which can be dropped from the problem later, ρ is the cost of equity² used as a continuously compounded discount rate, whereas $\int_{t_0}^{t_f} e^{-\rho t} (\sum_{i \in \mathcal{S}} \pi_i^f D_i^f) dt$ is the net present value (NPV) of revenue. From familiarity with these dynamics, we may restate them for all $f \in \mathcal{F}$ as

$$\frac{dD_i^f}{dt} = \eta_i^f \left(\frac{y_i}{|\mathcal{F}|(t - t_0)} - \pi_i^f \right) \quad \forall i \in \mathcal{S} \quad (12)$$

$$\frac{dy_i}{dt} = \sum_{g \in \mathcal{F} - \{f\}} \pi_i^g \quad \forall i \in \mathcal{S} \quad (13)$$

$$D_i^f(t_0) = K_{i,0}^f \quad \forall i \in \mathcal{S} \quad (14)$$

$$y_i(t_0) = 0 \quad \forall i \in \mathcal{S} \quad (15)$$

As a consequence we may rewrite the optimal control problem of firm $f \in \mathcal{F}$ as

²It is used rather than cost of capital for two main reasons. The first one is pragmatic as it yields the same cost of equity for any ATSP because this cost is only dependent on the systematic risk and market risk premium as suggested by the CAPM. Secondly, the cost of capital is affected by leverage, distressing and agency costs which are different across ATSP.

$$\max_{\pi^f} J_f(\pi^f, \pi^{-f}, t) = \int_{t_0}^{t_f} e^{-\rho t} \left(\sum_{i \in \mathcal{S}} \pi_i^f D_i^f \right) dt \quad (16)$$

Subject to

$$\frac{dD_i^f}{dt} = \eta_i^f \left(\frac{y_i}{|\mathcal{F}|(t - t_0)} - \pi_i^f \right) \quad \forall i \in \mathcal{S} \quad (17)$$

$$\frac{dy_i}{dt} = \sum_{g \in \mathcal{F} - \{f\}} \pi_i^g \quad \forall i \in \mathcal{S} \quad (18)$$

$$D_i^f(t_0) = K_{i,0}^f \quad \forall i \in \mathcal{S} \quad (19)$$

$$y_i(t_0) = 0 \quad \forall i \in \mathcal{S} \quad (20)$$

$$\pi_{min,i}^f \leq \pi_i^f \leq \pi_{max,i}^f \quad \forall i \in \mathcal{S} \quad (21)$$

$$D_i^f \geq 0 \quad \forall i \in \mathcal{S} \quad (22)$$

$$0 \leq AD_i^f \leq C_i^f \quad \forall i \in \mathcal{S} \quad (23)$$

Consequently,

$$D(\pi) = arg \left\{ \begin{array}{l} \frac{dD_i^f}{dt} = \eta_i^f \left(\frac{y_i}{|\mathcal{F}|(t - t_0)} - \pi_i^f \right), \frac{dy_i}{dt} = \sum_{g \in \mathcal{F} - \{f\}} \pi_i^g, \\ D_i^f(t_0) = K_{i,0}^f, D_i^f \geq 0, 0 \leq AD_i^f \leq C_i^f \quad \forall i \in \mathcal{S}, f \in \mathcal{F} \end{array} \right\} \quad (24)$$

Where we implicitly assume that the dynamics have solutions for all feasible controls. In compact notation this problem can be expressed as: with π^{-f} as exogenous inputs, compute π^f in order to solve the following optimal control problem:

$$\max_{\pi^f} J_f(\pi^f, \pi^{-f}, t) \text{ subject to } \pi^f \in \Theta_f \quad \forall f \in \mathcal{F} \quad (25)$$

Where $\Theta_f = \{\pi^f: (17) - (23) \text{ hold}\}$

So far, we have introduced a model for ATSP which is deterministic. However, in reality, model parameters are usually unknown to the modeler and the firm and follow probability distributions. Let us suppose the sales season or the total planning horizon is $[T_0, T_F]$, and we want to update model parameters L times within the season. Hence, the time horizon $[T_0, T_F]$ is divided into the following L sub-intervals:

$$[T_0, T_0 + \Delta T], [T_0 + \Delta T, T_0 + 2\Delta T], \dots, [T_0 + (L - 1)\Delta T, T_F]$$

Where $\Delta T = \frac{T_F - T_0}{L}$. At time $t = T_0 + l\Delta T$, each firm updates model parameters based on observations from the interval $[T_0 + (l-1)\Delta T, t = T_0 + l\Delta T]$, for $l = 1, 2, \dots, L-1$. Assuming that modeling error and observation errors follow normal distributions, we may employ Kalman filtering to estimate parameters. For Kalman filtering we minimize the squared estimation error. The posterior estimates are updated as long as the new observations are available. For most other methods, estimation is conducted only once and ignores the rich information contained in new observations. Also, the Kalman filter is robust in the sense that it can handle substantially inaccurate observations. Other methods usually assume that the observations are essentially accurate. For brevity we drop the superscript f for each firm. The dynamics for the demand are then

$$\frac{dD_i(t)}{dt} = \eta_i \left(\frac{\int_{t_0}^t \sum_{g \in \mathcal{F} - \{f\}} \pi_i^g(\tau) d\tau}{t - t_0} - \pi(t) \right) \quad \forall i \in \mathcal{S} \quad (26)$$

After one planning period is completed, we update the model parameter, η_i , to have a more précising policy for the next planning horizon. For this purpose, we collect data during the previous planning horizon. Although, we decide the pricing policy in continuous time, most data collecting activities typically occur in discrete time in practice. Let us employ the discrete time index k and assume we collect data K times within each planning period. Using the vector notation $\eta = (\eta_i: i \in \mathcal{S})$, the dynamics of η may be expressed as

$$\eta(k+1) = \eta(k) + \xi(k)$$

Where $\xi(k)$ is a random noise from a normal distribution $N(0, Q)$. The matrix Q is called the process-noise covariance matrix.

The value of the parameter η cannot be observed directly but rather only by the change in realized demand, may be observed which can be defined as

$$z(k) \equiv \Delta D(k) = \eta(k) \left(\frac{\sum_{j=0}^k \sum_{g \in \mathcal{F} - \{f\}} \pi_i^g(j)}{k \Delta k} - \pi(k) \right) \Delta k + \omega(k) \quad (27)$$

Where

$$\Delta D(k) = D(k+1) - D(k), \quad D = (D_i: i \in \mathcal{S}), \quad \Delta k \equiv \frac{t_f - t_0}{K}$$

And $\omega(k)$ is a random observation noise from a normal distribution $N(0, R)$. The matrix R is called the measurement noise covariance matrix. Expression (27) is obtained by discretizing the state dynamics (26) of D .

According to Bryson and Ho (1975), the Kalman filter dynamics are

$$\hat{\eta}(k) = \bar{\eta}(k) + V(k)[z(k) - H(k)\bar{\eta}(k)]$$

$$\bar{\eta}(k+1) = \hat{\eta}(k)$$

$$P(k) = [M(k)^{-1} + H(k)^T R^{-1} H(k)]^{-1}$$

$$M(k+1) = P(k) + Q$$

Where we defined

$$V(k) \equiv P(k)H(k)R^{-1}$$

$$H(k) \equiv \left(\frac{\sum_{j=0}^k \sum_{g \in \mathcal{F}} \pi_i^g(j)}{k \Delta k} - \pi(k) \right) \Delta k$$

Where $\bar{\eta}(k)$ is an *a priori* estimate of $\eta(k)$ before observation and $\hat{\eta}(k)$ is an *a posteriori* estimate after observation. Hence, the estimation process is:

1. From previous planning period obtain sequences of exercised prices $\{\pi(k)\}$ and observe changes in demand $\{z(k)\}$.
2. Let the time index $k = 0$ and assume the initial values $P(0) = 1, \bar{\eta}(0) = \eta_{previous}$.
3. For index k , forecast according to

$$\bar{\eta}(k+1) = \hat{\eta}(k)$$

$$M(k+1) = P(k) + Q$$

4. Based on the observation $z(k+1)$, update according to

$$\hat{\eta}(k+1) = \bar{\eta}(k+1) + V(k+1)[z(k+1) - H(k+1)\bar{\eta}(k+1)]$$

Where

$$H(k+1) \equiv \left(\frac{\sum_{j=0}^{k+1} \sum_{g \in \mathcal{F} - \{f\}} \pi_i^g(j)}{(k+1) \Delta k} - \pi(k+1) \right) \Delta k$$

$$P(k+1) = [M(k+1)^{-1} + H(k+1)^T R^{-1} H(k+1)]^{-1}$$

$$V(k+1) \equiv P(k+1)H(k+1)R^{-1}$$

5. If $k = K$ stop; otherwise set $k = k+1$ and go to step 3.

When we finish the estimation process, we have $\bar{\eta}(K)$ at hand, which is the value of η that will be used in the next period.

Before we proceed to the game-theoretic model for competition among ATSP, we first describe the so-called gradient projection algorithm, for the case when only one firm exists. We obtain an optimal control problem, in which the criterion is non-linear, the state dynamics are linear, and the control set is convex and compact with a state-space constraint (5). Although, the gradient projection algorithm is very popular among optimal control researches, it is not easy to find a written statement of the algorithm in economics literature. More

information, such as convergence and varieties of the method, is found in Minoux (1986), Polak (1971) and Bryson (1999).

To continue we penalize the state-space constraint so that the criterion becomes

$$\begin{aligned} \max_{\pi^f} J_f(\pi^f, \pi^{-f}, t) \\ = \int_{t_0}^{t_f} \left[e^{-\rho t} \left(\sum_{i \in S} \pi_i^f D_i^f \right) - \frac{\mu}{2} \left\{ \max(0, AD^f - C^f)^2 + \min(0, D^f)^2 \right\} \right] dt \quad (28) \end{aligned}$$

Where μ is the penalty coefficient. Thus, our problem is reduced to the following general class of optimal control problems:

$$\min J(u) = K[x(t_f), t_f] + \int_{t_0}^{t_f} f_0(x, u, t) dt \quad (29)$$

Subject to

$$\frac{dx}{dt} = f(x, u, t) \quad (30)$$

$$x(t_0) = x^0 \quad (31)$$

$$u \in U \subset V \quad (32)$$

Where x^0 is a known, fixed vector and both t_0 and t_f are fixed. Also, the definition of f_0 is implicit, based on Eq.(28). In addition, V is a Hilbert space; in particular, $V = (L^2[t_0, t_f])^m$. Note also that

$$H(x, u, \lambda, t) = f_0(x, u, t) + \lambda^T f(x, u, t)$$

Will be the Hamiltonian for the unconstrained problem (29)-(32).

According to this, the gradient projection algorithm can be summarized through the following steps:

1. Initialization. Set $k = 0$ and pick $u^0(t) \in (L^2[t_0, t_f])^m$.
2. Find state trajectory. Using $u^k(t)$ solve the state initial value problem

$$\begin{aligned} \frac{dx}{dt} &= f(x, u^0, t) \\ x(t_0) &= x^0 \end{aligned}$$

And call the solution $x^k(t)$.

3. Find adjoint trajectory. Using $u^k(t)$ and $x^k(t)$ solve the adjoint final value problem

$$(-1) \frac{d\lambda}{dt} = \frac{\partial H(x^k, u^k, \lambda, t)}{\partial x}$$

$$\lambda(t_f) = \frac{\partial K[x(t_f), t_f]}{\partial x}$$

and call the solution $\lambda^k(t)$.

4. Find gradient. Using $u^k(t)$, $x^k(t)$, and $\lambda^k(t)$ calculate

$$\nabla_u J(u^k) = \frac{\partial H(x^k, u^k, \lambda, t)}{\partial u} = \frac{\partial f_0(x^k, u^k, t)}{\partial u} + (\lambda^k)^T \frac{\partial f(x^k, u^k, t)}{\partial u}$$

5. Update and apply stopping test. For a suitably small step size ε_k , update according to

$$u^{k+1} = P_U[u^k - \varepsilon_k \nabla_u J(u^k)]$$

Where $P_U[\cdot]$ denotes the minimum projection onto U . if an appropriate stopping test is satisfied, declare $u^* \approx u^{k+1}(t)$ otherwise set $k = k + 1$ and go to step 2.

4. SOLVING THE GAME

Each ATSP is a Cournot-Nash agent that knows and employs the current instantaneous values of the decision variables of other firms to make its own non-cooperative decisions. Therefore, (25) defines a set of coupled optimal control problems, one for each firm $f \in \mathcal{F}$. It is useful to note that (25) is an optimal control problem with fixed terminal time and fixed terminal state. Its Hamiltonian is

$$H_f(\pi^f; D^f; \lambda^f; \sigma; \alpha^f; \beta^f; \pi^{-f}; t)$$

$$\equiv e^{-\rho t} \left(\sum_{i \in \mathcal{S}} \pi_i^f D_i^f \right) + \Lambda_f(\pi^f; D^f; \lambda^f; \sigma^f; \alpha^f; \beta^f; \pi^{-f}) \quad (33)$$

Where

$$\Lambda_f(\pi^f; D^f; \lambda^f; \sigma^f; \alpha^f; \beta^f; \pi^{-f})$$

$$= \left\{ \sum_{i \in \mathcal{S}} \lambda_i^f \left[\eta_i^f \left(\frac{y_i}{|\mathcal{F}|(t-t_0)} - \pi_i^f \right) \right] + \sum_{i \in \mathcal{S}} \sigma_i \left(\pi_i^f + \sum_{g \in \mathcal{F}-\{f\}} \pi_i^g \right) \right.$$

$$\left. - \sum_{i \in \mathcal{S}} \alpha_i^f D_i^f + \sum_{i \in \mathcal{S}} \beta_i^f (AD_i^f - C_i^f) \right\} \quad (34)$$

While $\lambda_i^f \in \mathcal{H}^1[t_0, t_f]$ is the adjoint variable for the dynamics associated with the firm f with service type i while $\lambda \in (\mathcal{H}^1[t_0, t_f])^{|\mathcal{S}| \times |\mathcal{F}|}$, $\sigma_i \in \mathbb{R}_+^1$, $\alpha_i \in \mathbb{R}_+^1$ and $\beta_i \in \mathbb{R}_+^1$ are the dual variables arising from the auxiliary state

variables $(y_i, i \in \mathcal{S})$ and state-space constraints before estimation. The instantaneous revenue for firm f is $\sum_{i \in \mathcal{S}} \pi_i^f D_i^f$. We assume in the balance of this paper that the game (25) is regular in the sense of Definition 1. Therefore, the maximum principle (Bryson and Ho, 1975) tells us that an optimal solution to (25) is a sextuplet $\{\pi^{f*}(t), D^{f*}(t); \lambda^{f*}(t), \sigma^*, \alpha^{f*}, \beta^{f*}\}$ requiring that the non-linear program

$$\max H_f \text{ subject to } \pi_{min}^f \leq \pi^f \leq \pi_{max}^f$$

Be solved by every firm $f \in \mathcal{F}$ for every instant of time $t \in [t_0, t_f]$ where

$$\pi_{min}^f = \{\pi_i^f : i \in \mathcal{S}\}$$

$$\pi_{max}^f = \{\pi_i^f : i \in \mathcal{S}\}$$

Consequently, any optimal solution must satisfy at each time $t \in [t_0, t_f]$

$$\pi^{f*} = \arg \left\{ \max_{\pi_{min}^f \leq \pi^f \leq \pi_{max}^f} H_f(\pi^f; D^f; \lambda^f; \sigma; \alpha^f; \beta^f; \pi^{-f}; t) \right\} \quad (35)$$

Which in turn, by virtue of regularity, is equivalent to

$$[\nabla_{\pi^f} H_f^*]^T (\pi^f - \pi^{f*}) \leq 0 \quad \forall \left(\frac{\pi_{min}^f}{\pi_{max}^f} \right) \leq \left(\frac{\pi^{f*}}{\pi^f} \right) \leq \left(\frac{\pi_{max}^f}{\pi_{min}^f} \right) \quad (36)$$

Where

$$H_f^* \equiv e^{-\rho t} \left(\sum_{i \in \mathcal{S}} \pi_i^{f*} D_i^{f*} \right) + \Lambda_f(\pi^{f*}; D^{f*}; \lambda^{f*}; \sigma^{f*}; \alpha^{f*}; \beta^{f*}; \pi^{-f}) \quad (37)$$

From (33)

$$\left\{ \nabla_{\pi^f} \left[e^{-\rho t} \left(\sum_{i \in \mathcal{S}} \pi_i^{f*} D_i^{f*} \right) \right] + \Lambda_f(\pi^{f*}; D^{f*}; \lambda^{f*}; \sigma^{f*}; \alpha^{f*}; \beta^{f*}; \pi^{-f}) \right\}^T \times (\pi^f - \pi^{f*}) \leq 0 \quad \forall \left(\frac{\pi_{min}^f}{\pi_{max}^f} \right) \leq \left(\frac{\pi^{f*}}{\pi^f} \right) \leq \left(\frac{\pi_{max}^f}{\pi_{min}^f} \right)$$

Furthermore, adjoint dynamics and state dynamics are

$$\frac{\partial H_f}{\partial D_i^f} = (-1) \frac{dy_i^{f*}}{dt} \quad (38)$$

$$\frac{\partial H_f}{\partial \lambda_i^f} = \frac{dD_i^{f*}}{dt} \quad (39)$$

Due to absence of terminal time constraint, the transversality condition is

$$\lambda^{f*}(t_f) = \gamma^T \frac{\partial \Gamma[D^{f*}(t_f), t_f]}{\partial D^{f*}(t_f)} = 0$$

Which gives rise to a two point boundary value problem.

With this preceding background, we are now in a position to create a variational inequality for non-cooperative competition among the firms. We consider the following DVI which has solutions that are Cournot-Nash equilibria for the game described above in which individual ATSP maximize their revenue in light of current information about their competitors: find $\pi^* \in \Xi$ such that

$$\int_{t_0}^{t_f} \left(\sum_{i \in S} \sum_{f \in F} \frac{\partial H_f^*}{\partial \pi_i^f} (\pi^f - \pi^{f*}) \right) dt \quad \forall \pi \in \Xi = \prod_{f \in F} \Xi_f \quad (40)$$

Where H_f^* is defined in (37) and $\Xi_f = \{\pi^f : \pi_{min}^f \leq \pi^f \leq \pi_{max}^f\}$

This DVI is regular in the sense of Definition 1 and a convenient way of expressing the Cournot-Nash game that is our present interest.

When the regularity conditions given in Definition 1 holds, $DVI(F, f, U)$ is reduced to the class of variational inequalities in Hilbert space considered by Konnov et al. (2002) for which U is a non-empty closed and convex subset of a Hilbert space, and F is a continuously differentiable mapping of u . This allows us to analyze $DVI(F, f, U)$ by considering gap functions, which are defined as follows:

Definition 2. A function $\delta: U \rightarrow \mathbb{R}_+$ is called a gap function for $DVI(F, f, U)$ when the following statements hold:

1. $\delta(u) \geq 0 \quad \forall u \in U$
2. $\delta(u) = 0$ if and only if u is the solution of $DVI(F, f, U)$

Let us now consider the function

$$\delta_\alpha(u) = \max_{v \in U} \Phi_\alpha(u, v) \quad (41)$$

Where

$$\Phi_\alpha(u, v) = \langle F(x, u, t), u - v \rangle - \alpha \phi(u, v)$$

$$x(u, t) = \arg \left\{ \frac{dy}{dt} = f(y, u, t), y(t_0) = y^0 \right\} \in (\mathcal{H}^1[t_0, t_f])^n, U \subseteq (L^2[t_0, t_f])^m$$

We assume ϕ is a function that satisfies following assumptions: (1) ϕ is continuously differentiable on $(L^2[t_0, t_f])^{2m}$; (2) ϕ is non-negative on $(L^2[t_0, t_f])^{2m}$; (3) $\phi(u, \cdot)$ is strongly convex with modulus $c > 0$ for any $u \in (L^2[t_0, t_f])^m$; and (4) $\phi(u, v) = 0$ if and only if $u = v$. The maximization problem (41) has a unique solution since $\Phi_\alpha(u, v)$ is strongly convex in v and U is convex. Let $v_\alpha(u)$ be the solution of (41) such that $\delta_\alpha(u) = \Phi_\alpha(u, v_\alpha(u))$. Then, we have the following result from Kwon and Friesz (2007):

Lemma. The function $\delta_\alpha(u)$ defined by (41) is a gap function for $DVI(F, f, U)$, and u is the solution to $DVI(F, f, U)$ if and only if $u = v_\alpha(u)$.

In addition to the regular gap function, let us introduce a special gap function, the so-called D-gap functions, which has the form

$$\psi_{\alpha\beta}(u) = \delta_\alpha(u) - \delta_\beta(u)$$

For $0 < \alpha < \beta$. While $\delta_\alpha(u)$ is not differentiable in general, $\psi_{\alpha\beta}(u)$ is Gateaux-differentiable. Among many gap functions, Fukushima (1992) considered a case of gap function, the regularized gap function, and Konnov *et al.* (2002) extended it to Hilbert spaces. In particular we note

$$\phi(u, v) = \frac{1}{2} \|v - u\|^2 \quad (42)$$

Is strongly convex with modulus $c > 0$. Adopting Fukushima's (1992) gap function, we obtain

$$\Phi_\alpha(u, v) = \langle F(x, u, t), u - v \rangle - \frac{\alpha}{2} \|v - u\|^2$$

The corresponding D-gap function becomes

$$\begin{aligned} \psi_{\alpha\beta}(u) &= \langle F(x, u, t), v_\beta(u) - v_\alpha(u) \rangle - \frac{\alpha}{2} \|v_\alpha(u) - u\|^2 \\ &\quad + \frac{\beta}{2} \|v_\beta(u) - u\|^2 \end{aligned} \quad (43)$$

Where we denote

$$v_\alpha(u) = \arg \max_{v \in U} \Phi_\alpha(u, v) \quad (44)$$

$$v_\beta(u) = \arg \max_{v \in U} \Phi_\beta(u, v) \quad (45)$$

With the D-gap function (43), $DVI(F, f, U)$ is equivalent to the following optimal control problem $OCP(\psi_{\alpha\beta}, f, U)$:

$$\begin{aligned}
 \min \psi_{\alpha\beta}(u) &= \langle F(x, u, t), v_{\beta}(u) - v_{\alpha}(u) \rangle - \frac{\alpha}{2} \|v_{\alpha}(u) - u\|^2 \\
 &\quad + \frac{\beta}{2} \|v_{\beta}(u) - u\|^2 \\
 &= \int_{t_0}^{t_f} \left\{ F(x, u, t) [v_{\beta}(u) - v_{\alpha}(u)] - \frac{\alpha}{2} [v_{\alpha}(u) - u]^2 \right. \\
 &\quad \left. + \frac{\beta}{2} [v_{\beta}(u) - u]^2 \right\} dt \quad (46)
 \end{aligned}$$

Subject to

$$\frac{dx}{dt} = f(x, u, t) \quad (47)$$

$$x(0) = x_0 \quad (48)$$

$$u \in U \quad (49)$$

This is a Bolza form of standard optimal control problem except that the objective functional involves the maximization of sub-problems defined by (44) and (45).

Now, we are interested in the gradient of the objective functional $\psi_{\alpha\beta}(u)$, which is, in fact, equivalent to the gradient of the corresponding Hamiltonian for $OCP(\psi_{\alpha\beta}, f, U)$:

$$\begin{aligned}
 H(x, u, \lambda, t) &= F(x, u, t) [v_{\beta}(u) - v_{\alpha}(u)] - \frac{\alpha}{2} [v_{\alpha}(u) - u]^2 + \frac{\beta}{2} [v_{\beta}(u) - u]^2 \\
 &\quad + \lambda f(x, u, t) \quad (50)
 \end{aligned}$$

The gradient of $\psi_{\alpha\beta}$ is determined as follows

Theorem³ Suppose $F(x, u, t)$ is Lipschitz continuous on every bounded subset of $(L^2[t_0, t_f])^m$. Then $\psi_{\alpha\beta}(u)$ is continuously differentiable in the sense of Gateaux, and

$$\begin{aligned}
 \nabla \psi_{\alpha\beta}(u) &= \frac{\partial}{\partial u} H(x, u, \lambda, t) \\
 &= \frac{\partial F(x, u, t)}{\partial u} [v_{\beta}(u) - v_{\alpha}(u)] + \alpha [v_{\alpha}(u) - u] \\
 &\quad - \beta [v_{\beta}(u) - u] + \lambda \frac{\partial f(x, u, t)}{\partial u}
 \end{aligned}$$

We propose the following descent algorithm for $OCP(\psi_{\alpha\beta}, f, U)$, in which the main philosophy remains same as in usual optimal control problems except

³The proofs of theorems in this section and detail discussions for gap functions are found in Kwon and Friesz (2007).

we need to solve the state dynamics, the adjoint dynamics and two sub-problems for v_α and v_β to obtain the current descent direction:

1. Initialization. Choose $0 < \alpha < \beta$. Pick $u^k(t) \in U, k = 0$.
2. Find state variables. Solve the state dynamics

$$\begin{aligned}\frac{dx}{dt} &= f(x, u^k, t) \\ x(0) &= x_0\end{aligned}$$

Call the solution $x^k(t)$.

3. Find adjoint variables. Solve the adjoint dynamics

$$\begin{aligned}(-1)\frac{d\lambda}{dt} &= \nabla_x H(x^k, u^k, \lambda, t) \\ &= \frac{\partial F(x^k, u^k, t)}{\partial x} [v_\beta(u^k) - v_\alpha(u^k)] + \lambda \frac{\partial f(x^k, u^k, t)}{\partial x} \\ \lambda(t_f) &= 0\end{aligned}$$

and call the solution $\lambda^k(t)$ where

$$\begin{aligned}v_\alpha(u^k) &= P_U \left[u^k - \frac{1}{\alpha} F(x^k, u^k, t) \right] \\ v_\beta(u^k) &= P_U \left[u^k - \frac{1}{\beta} F(x^k, u^k, t) \right]\end{aligned}$$

4. Find the gradient. Determine

$$\begin{aligned}\nabla \psi_{\alpha\beta}^k(t) &= \nabla_u H(x^k, u^k, \lambda^k, t) \\ &= \frac{\partial F(x^k, u^k, t)}{\partial u} [v_\beta(u^k) - v_\alpha(u^k)] + \alpha [v_\alpha(u^k) - u^k] \\ &\quad - \beta [v_\beta(u^k) - u^k] + \lambda \frac{\partial f(x^k, u^k, t)}{\partial u}\end{aligned}$$

5. Update the current control. For a suitably small step size $\theta_k \in \mathbb{R}_{++}^1$ determine $u^{k+1}(t) = P_U[u^k(t) - \theta_k \nabla \psi_{\alpha\beta}^k(t)]$

6. Stopping test. For $\epsilon \in \mathbb{R}_{++}^1$, a preset tolerance, stop if $\|\psi_{\alpha\beta}(u^{k+1})\| < \epsilon$ and declare $u^* \approx u^{k+1}$ otherwise set $k = k + 1$ and go to Step 2.

We state the following convergence result without proof:

Theorem of convergence. Suppose the functional $\psi_{\alpha\beta}: U \rightarrow \mathbb{R}_+$ is strongly convex with modulus $c > 0$ and $\nabla \psi_{\alpha\beta}(u)$ is defined and satisfies the Lipschitz condition $\|\nabla \psi_{\alpha\beta}(u_1) - \nabla \psi_{\alpha\beta}(u_2)\| \leq \omega \|u_1 - u_2\| \forall u_1, u_2 \in U$. Then the descent algorithm converges to the minimum u^* of $\psi_{\alpha\beta}$ on U for step size choices $\theta \in (0, \frac{2c}{\omega^2})$.

5. APPLICATION

Algerian market for telecommunication services is governed by three firms specifically the historical public held company "Algerie Telecom, AT" through its commercial brand "Mobilis", "Orascom Telecom Algerie, OTA" known by the commercial name "Djeezy" (private owned company, division of Orascom Holding, Egypt), and "Wataniya Telecom Algerie, WTA" under the name "Nedjma" (private owned company, division of Qtel, Qatar). In fact, the market of telecommunications in Algeria was archaic prior the existence of OTA leader of the market with more than 13 million clients. OTA was incorporated as an Algerian company in July 2001 after paying for a GSM license of \$737 million. OTA relied heavily on technologies from Alcatel and Siemens to invest between 2001 and 2006 more than \$2.7 billion. OTA has 3000 employees and represents almost 45% from total revenues of Orascom Holding. Since November 2009, problems has been started to appear between Algerian government and OTA through \$596.6 million of tax adjustment for years 2005-2007 and the problem tended to the zenith at the end of 2010 when the Algerian government exhibited officially its intentions to repurchase OTA. Actually, OTA is still a division of Orascom Holding which was recently acquired by a Russian group. Meanwhile, OTA's evaluation is under processing by consulting firms designated by Algerian government.

The second private firm, WTA recorded its first positive profits in 2010 since its introduction to the Algerian market in 2004 for a GSM license of \$421 million. In fact, WTA realized a net profit of \$10.1 million in 2010 versus a negative net profit of \$34.1 million in 2009. Earnings before Interests, Taxes, and Depreciation (EBITDA) are \$230.4 million in 2010 versus \$161.7 million in 2009 with a growth of 42.5%. WTA's revenues grow from \$491.7 million in 2009 to \$610.4 million in 2010 for 8.2 million clients at the end of 2010 to demonstrate a 30% of the Algerian market shares with average revenue per client (APRU) of \$6.4 for the year 2010. WTA invested at least \$1.5 billion since its introduction to the Algerian market.

Non available data by type of service and evolution vis-a-vis competitor prices constrained the application to be carried out on a calibrated example that may substantially represent the reality of Algerian market for telecommunications. The author attempted to rely on pragmatic rules in selecting hypothetical values to be as closer as possible from the reality. A calibration technique was used to simulate the data of the application. Global parameters of the calibration were statistically estimated on the basis of collected data from official financial publications of WTA and OTA official web portals. Data for AT was basically collected from various sources encompassing AT's annual publications, national press releases, and internal reports from AT headquarters. Due to ethical considerations and privacy data

obligations, the identity of firms was kept secret and we stick the application using the mathematical notation "Firm f ".

The descent method described earlier converged after 18 iterations for this application with the preset tolerance $\varepsilon = 10^{-6}$ for gap function. The computer code for the descent method is written in MATLAB 7.0. The run time for this application is less than 20 seconds using a laptop with an Intel Pentium Dual CPU 1.73 GHz processor and 2 GB RAM.

Let us consider the Algerian market where three ATSP $f = 1, 2, 3$ are offering four services $i = 1, 2, 3, 4$. The planning horizon for this application is a month or 30 days. Each firm updates parameters once daily based on demand observations. The speed of change in demand η takes on the values below

η	Service type, i	1	2	3	4
	Firm 1	0.12	0.10	0.14	0.11
	Firm 2	0.11	0.07	0.10	0.15
	Firm 3	0.10	0.08	0.12	0.09

Each firm's initial demand K_0 (values are in million units) at time t_0 for services i is shown below

K_0	Service type, i	1	2	3	4
	Firm 1	10.5	18.0	23.0	30.5
	Firm 2	9.5	16.5	20.0	31.0
	Firm 3	10.0	17.5	22.5	30.0

The resource upper bound for each firm is given below

C	Resource type, i	1	2	3	4	5
	Firm 1	330	240	180	90	85
	Firm 2	180	150	120	75	10
	Firm 3	300	210	150	60	255

Assume the bounds on prices π_{max} and π_{min} (values are in \$ using an exchange rate of \$1=75DA) are those given below

π_{max}	Service type, i	1	2	3	4
	Firm 1	0.50	85.00	100.00	55.00
	Firm 2	0.39	67.50	42.70	21.20
	Firm 3	0.45	77.89	52.17	34.50
π_{min}	Service type, i	1	2	3	4
	Firm 1	0.01	8.50	10.00	5.50
	Firm 2	0.01	6.75	4.27	2.12
	Firm 3	0.01	7.80	5.22	3.45

The demand data for the different services was simulated on the basis of a Brownian motion diffusion process that starts with initial demands K_0 . This kind of simulation is used in financial applications especially to predict stock prices and derivatives. Using all this data and assuming a unit matrix for the incidence matrix (i.e., each service is using a part of available resources), ATSP determine an optimal pricing policy (Table 2) by the mean of the developed descent algorithm for the game that was established in this paper. Once ATSP have determined an optimal pricing policy, they exercise it and observe what really happens in the market. The observation of the actual realized demand occurs every week so that the firm has 52 observations for the year. With this data, ATSP learn the model parameter η when the process-noise covariance and the measurement noise covariance matrices are the following:

$$Q = \begin{bmatrix} 0.3 & 0.1 & 0.2 & 0.2 \\ 0.1 & 0.5 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.2 & 0.3 \end{bmatrix}, R = \begin{bmatrix} 0.2 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.3 \end{bmatrix}$$

Table 1. ATSP Revenues

Revenues (by \$ million)	Firm 1	Firm 2	Firm 3
Before estimation	605	435	501
Observed	512	395	417
After estimation	410	289	307

Table 2. ATSP's Optimal prices

Service type, i	1	2	3	4
π^* Firm 1	0.031	8.610	9.954	5.629
Firm 2	0.042	6.778	4.285	2.142
Firm 3	0.047	7.923	5.238	3.464

As the results presented in Table 1, the observed revenue is lower than the projected *a priori* best revenue estimation. This means that the actual speed of change in demand is faster than used in planning. The firm may decrease price but will not have the ability to raise its demand. This result seems interesting as it can provide empirical evidence that supports oligopolistic equilibrium of the Algerian market for 1 unit call from any ATSP around 4 DA without being motivated to deviate significantly from this value. With the Kalman filter dynamics, ATSP adjust the parameter η according to

$$\eta_{adjusted} = \begin{bmatrix} 0.1203 \\ 0.1236 \\ 0.1575 \\ 0.1271 \end{bmatrix}$$

Whose values are larger than the original values. With this new forecast, ATSP perform an analysis for the next period. On the basis of results in Tables

1 and 2 it can be inferred that prices are relatively lower for the simulation after estimation than before estimation.

We note that the price will not reach a stationary state, in general. The optimal price trajectories ATSP would obtain in each planning period are up-to-date reflections of customers' choice behavior; hence the price trajectories will depend on the current market state which is uncertain. However, if there is no shock to the market and ATSP have a long enough time horizon, the price trajectory must reach a stationary state.

6. CONCLUSION

It has been shown that non-cooperative competition among service ATSP who engage in demand learning may be modeled as a differential variational inequality involving Kalman filtering. It has been also shown that such a model may be effectively solved using the notion of a gap function together with descent in Hilbert space. Indeed the performance of the gap function/descent algorithm we propose suggests that revenue optimization/pricing calculations for real market involving oligopolistic competition among ATSP who are Nash agents engaged in demand learning may be carried out and used for decision support. The most obvious extension of the modeling framework proposed herein is to create a differential Stackelberg leader-follower game wherein the followers are described by a differential variational inequality like the one we have presented. The leader of such a game will be a powerful telecommunication service (e.g., Orange Telecom) that seeks to enter the market and who is described by an appropriate optimal control problem for revenue maximization.

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